

The black-hole/qubit correspondence: an up-to-date review

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We give a review of the black-hole/qubit correspondence that incorporates not only the earlier results on black hole entropy and entanglement measures, seven qubits and the Fano plane, wrapped branes as qubits and the attractor mechanism as a distillation procedure, but also newer material including error-correcting codes, Mermin squares, Freudenthal triples and 4-qubit entanglement classification.

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1. INTRODUCTION

Six years have passed since the realization [1–3] that the apparently separate fields of quantum information [4] and string theory can be related. When two different branches of theoretical physics share many of the same features, it frequently allows knowledge on one side to provide new insights on the other. This has certainly proved to be the case with string theory and quantum information as we hope to show in this review. The original correspondence was between the structure of the Bekenstein-Hawking entropy [5, 6] formulae of certain black hole solutions in string theory, and certain multipartite entanglement measures [7] in quantum information. However, many more striking structural similarities between these fields have since been uncovered forming what has become known as the “black-hole/qubit correspondence” (BHQC).

As far as the BHQC is concerned the main reason for the occurrence of these striking coincidences seems to be the presence of similar symmetry structures. Namely, within the field of stringy black holes there are the U-duality groups [8–11], for a review see [12], and in the field of multipartite entanglement the groups of admissible local manipulations preserving entanglement type [13, 14]. While in the former case the groups in question are real (in the supergravity approximation) and in the latter they are complex, in particular instances a suitable complex extension places these stringy dualities in the realm of local multipartite entanglement transformations. Then the U-duality invariants can be mapped to invariants (entanglement measures) under the local group of admissible manipulations. Then it is not so surprising that the most obvious invariants that can show up in these different scenarios are U-duality invariant formulae of the black hole entropy.

This realization paved the way for studying the structure of black-hole entropy formulae via the techniques of entanglement measures [1–3, 15–17] and three-qubit Pauli operators [18, 19] or, conversely, getting hints for constructing new and physically interesting measures from the structure of entropy formulae [20, 21]. Another useful aspect of this correspondence is that the classification problem of certain types of black hole can be mapped to the classification problem of entanglement types of qubit systems [2, 3, 21–23].

Apart from clarifying the structure of black-hole entropy formulae, there has been some progress in understanding the dynamical aspects of the moduli, in particular the famous attractor mechanism, in entanglement terms [24–26]. It is known that in general the entropy of a black hole can depend on the values of moduli or scalar fields, having their origin in the compactification of extra dimensions. However, for the special case of extremal black holes, having zero Hawking temperature, the values of such scalar fields are fixed on the black hole horizon in terms of the charges. The crucial point is that the fixed values are independent of the asymptotic values of such moduli. The radial evolution of the scalar fields starting from the asymptotically Minkowski region to the horizon shows a fixed point behaviour. In the special case of the so called STU -model [27, 28] it has been shown that

such moduli stabilisation can be recast in the language of quantum information as a radial evolution of a three-qubit state depending on the charges, the moduli and the warp factor resulting in a distillation procedure of GHZ (Greenberger-Horne-Zeilinger) like states on the horizon [3, 29–32].

The BHQC has shed new light on well-known results in quantum entanglement [33] and has clarified previously controversial issues [22, 23]. The techniques employed in such cases have originated from the string theoretical side. For example, although Freudenthal triple systems (FTS) [34–36] have been well-known to the supergravity community [37–39], their relevance to special entangled systems [20, 21, 40] has only recently been realised within the framework of the BHQC. This, in turn, inspired new applications of the Freudenthal triple system to black holes in the form of Freudenthal and Jordan dualities [41, 42].

In some special cases multiqubit entanglement measures have turned out to be useful for classifying multicenter black hole solutions [43, 44]. These studies connected the structure of four-qubit invariants [45, 46] to the structure of elliptic curves and the j invariant [44]. The idea that such objects might play some role in four-qubit systems and the BHQC was first suggested in [47] and some related discussion also appeared in the supergravity [48] and quantum entanglement [49] literature.

The BHQC attempted to clarify the possible microscopic origin of qubits (and qutrits) in this entanglement based approach. It has been suggested that in the case of toroidal compactifications the appearance of qubits can be traced back to wrapping configurations of membranes [50] on noncontractible loops of the extra dimensions. This heuristic picture has been generalised and made more precise [51] by identifying the Hilbert space where these qubits reside within the cohomology of the extra dimensions. As a bonus it was also shown [51] that in this special case the phenomenon of flux compactification [52] can also be included within the realm of the BHQC.

The BHQC combined with the methods of finite geometry have provided a new conceptual framework for understanding the role of incidence geometries in quantum theory. It has been shown that the structure of certain black hole entropy formulae are encapsulated in the incidence structure of geometric hyperplanes [53] of finite geometries based on the two, three and four-qubit Pauli groups [18, 19]. Such groups have already made their debut to quantum error correcting codes [54], objects that have also been shown to play some role in the black hole attractor mechanism [16, 29, 55]. A surprising result is that Mermin squares [56] as geometric hyperplanes show up naturally in noncommutative parametrizations of incidence geometries characterising the structure of black hole entropy formulae. Such results initiated a further study of automorphism groups of finite geometric structures related to special subgroups of the U-dualities [18, 19, 57, 58], and a systematic study of the Veldkamp space of geometric hyperplanes for multiple qubits [57].

Though the BHQC is still at its infancy, it has repeatedly proved useful for obtaining interesting results on both sides of the correspondence by employing the techniques and methods of the other [3, 15–19, 22, 23, 29, 30, 32, 40, 44, 50, 51, 59–61]. Over the past six years joint efforts of two groups culminated in establishing a precise dictionary between the two sides of the correspondence [55, 59]. The aim of the present review is to give an account of these efforts.

2. CAYLEY'S HYPERDETERMINANT AND BLACK HOLE ENTROPY

2.1. Entanglement and three-qubit systems

Since entanglement may be used in the course of a quantum computation [4], characterising the “amount” of entanglement possessed by a given state is an important problem. There are several criteria for good measures of entanglement [7]. In particular, since entanglement is a global phenomenon of a quantum nature (in the sense that it leads to correlations between spatially separated systems that admit no classical explanation), any good measure should be monotonically decreasing under local operations (LO) on the constituent systems supplemented by classical communication (CC) between them [7, 13, 14]. LOCC operations cannot create entanglement. Hence, two states that may be stochastically (S) interrelated by an LOCC protocol have the same entanglement under any good measure [191]. Two states of a k -constituent composite system with Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$, $\dim \mathcal{H}_i = n_i$, are SLOCC-equivalent if and only if they are related by the subset of invertible local operation in SLOCC, i.e. elements of $\text{GL}(n_1, \mathbb{C}) \times \cdots \times \text{GL}(n_k, \mathbb{C})$ [14]. This SLOCC-equivalence group (which we will often refer to loosely as simply SLOCC) partitions the state space to entanglement classes. Any relative invariant of SLOCC is a good entanglement measure [62].

For three qubits we have three two-state systems each with Hilbert space \mathbb{C}^2 : \mathcal{H}_A , \mathcal{H}_B and \mathcal{H}_C where the labels refer to Alice, Bob and Charlie. The Hilbert space of the total system is $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. A three-qubit state of general form can be represented as

$$|\psi\rangle = \sum_{ABC=0,1} \psi_{ABC} |ABC\rangle, \quad |ABC\rangle = |A\rangle \otimes |B\rangle \otimes |C\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C. \quad (2.1)$$

Under SLOCC transformations our state transforms as a $(2, 2, 2)$ namely

$$|\psi\rangle \mapsto (\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C})|\psi\rangle, \quad \psi_{ABC} \mapsto \mathcal{A}_A^{A'} \mathcal{B}_B^{B'} \mathcal{C}_C^{C'} \psi_{A'B'C'}, \quad \mathcal{A}, \mathcal{B}, \mathcal{C} \in \text{GL}(2, \mathbb{C}). \quad (2.2)$$

Now one can define the quantity (Cayley's hyperdeterminant) [63, 64]

$$D(\psi) = [\psi_0\psi_7 - \psi_1\psi_6 - \psi_2\psi_5 - \psi_3\psi_4]^2 - 4[(\psi_1\psi_6)(\psi_2\psi_5) + (\psi_2\psi_5)(\psi_3\psi_4) + (\psi_3\psi_4)(\psi_1\psi_6)] + 4\psi_1\psi_2\psi_4\psi_7 + 4\psi_0\psi_3\psi_5\psi_6 \quad (2.3)$$

where $(\psi_0, \psi_1, \dots, \psi_7) \equiv (\psi_{000}, \psi_{001}, \dots, \psi_{111})$, which gives rise to a famous entanglement measure called the *three-tangle* [65] which for normalised states satisfies

$$0 \leq \tau_{ABC} = 4|D(\psi)| \leq 1. \quad (2.4)$$

Under SLOCC transformations $D(\psi)$ transforms as

$$D(\psi) \mapsto (\text{Det}\mathcal{A})^2(\text{Det}\mathcal{B})^2(\text{Det}\mathcal{C})^2 D(\psi) \quad (2.5)$$

hence this polynomial is a relative invariant. Notice that the expression of the three-tangle is invariant under permutations (triality) and the subgroup $[\text{SL}(2, \mathbb{C})]^{\otimes 3}$ of SLOCC transformations. The physical meaning of the three-tangle is the residual distributed entanglement not contained in either the pure state or the mixed state entanglement of any bipartite-singlepartite split [65]. Considerations of distributed entanglement have also been used in connection with attractors of *STU* black holes [31].

In this formalism the classification problem of entanglement types amounts to finding the $[\text{GL}(2, \mathbb{C})]^{\otimes 3}$ orbits of a particular $|\psi\rangle$. This problem has been solved by mathematicians [64] and later rediscovered by physicists [14]. The result is that apart from the trivial class with $|\psi\rangle = 0$ we have six SLOCC classes. The four classes that represent states with some degree of separability are the totally separable states with representative $|000\rangle$, the biseparable states with (unnormalized) representatives: $|0\rangle \otimes (|00\rangle + |11\rangle)$ and two similar states with the qubits cyclically permuted. Three qubits can be entangled in two inequivalent ways [14], the unnormalized representatives of these classes are the so-called $|GHZ\rangle$ and $|W\rangle$ states with the form

$$|GHZ\rangle = |000\rangle + |111\rangle, \quad |W\rangle = |001\rangle + |010\rangle + |100\rangle. \quad (2.6)$$

The important point is that these two classes can be separated from the rest as follows. The GHZ-class is characterised by $D(GHZ) \neq 0$, i.e. this state has nonvanishing three-tangle. On the other hand, it can be shown [33] that one can introduce a dual three-qubit state $|\tilde{\psi}\rangle$ which also transforms as a $(2, 2, 2)$ of $[\text{GL}(2, \mathbb{C})]^{\otimes 3}$. The dual state is cubic in the original amplitudes of $|\psi\rangle$ and its explicit expression [33] is connected to the so-called trilinear form of the corresponding Freudenthal triple system [33, 36]. Then one can show that the W -class is characterized by the conditions $D(W) = 0$, $|\tilde{\psi}\rangle \neq 0$. States that are having $D = 0$ and $|\tilde{\psi}\rangle = 0$ are either separable or biseparable. There is a nice geometric characterisation of these entanglement classes [3, 66, 67] in terms of twistors. Notice also that for states in the GHZ-class one can define the new state $|\hat{\psi}\rangle \equiv |\tilde{\psi}\rangle / \sqrt{|D(\psi)|}$ which is a special case of the Freudenthal dual state which plays an important role in the physics of black holes admitting a Freudenthal dual [41].

2.2. Three-qubit entanglement and Black Hole Entropy

An interesting subsector of string compactification to four dimensions is provided by the *STU* model. This model has a low energy limit which is described by $\mathcal{N} = 2$ supergravity coupled to three vector multiplets [27, 28, 68, 69]. This model can be obtained as a consistent truncation of different string theories in a number of ways. One possibility is to take the type IIA string theory compactified on a Calabi-Yau manifold and then consider a convenient truncation of the $\mathcal{N} = 2$ theory arising as the low energy limit. As an alternative possibility one can start with the heterotic string on the six torus T^6 . Then the *STU* model arises as a truncation of the resulting $\mathcal{N} = 4$ theory. The *STU* model got its name from the names of the three complex scalar fields (S, T and U), which play different roles in the different interpretations [27]. The *STU* model admits extremal black hole solutions carrying four electric and four magnetic charges. Within the framework of this model the macroscopic black hole entropy can be calculated [70].

The starting point of the black-hole/qubit correspondence was the observation that if we organize the eight charges of the solution into a $2 \times 2 \times 2$ array, i.e. a hypermatrix, for BPS (Bogomolnyi-Prasad-Sommerfield) solutions the macroscopic black hole entropy can be expressed as the negative of the square root of Cayley's hyperdeterminant [1]. The four electric (q_0, q_1, q_2, q_3) and four magnetic (p^0, p^1, p^2, p^3) charges, and the amplitudes of an *unnormalized* three-qubit state as follows

$$(p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3) \leftrightarrow (\psi_0, \psi_1, \psi_2, \psi_4, -\psi_7, \psi_6, \psi_5, \psi_3). \quad (2.7)$$

Then the macroscopic entropy is

$$S = \pi \sqrt{-D(\psi)} = \frac{\pi}{2} \sqrt{\tau_{ABC}(\psi)}. \quad (2.8)$$

From this expression we see that the entropy of such black holes can be related to a tripartite entanglement measure, namely the three-tangle. Notice, however, these are not three-qubit states, in the conventional sense. First of all, the amplitudes are *real* and unnormalized. Moreover, the charges should be also quantized, hence the amplitudes should be *integer*. To cap all this, for BPS solutions, for which half of the supersymmetry is conserved, $D(\psi)$ is negative [28].

However, the apparent issue of unnormalized states is not serious since SLOCC transformations do not preserve the norm in any case. As far as the reality of the amplitudes is concerned, one can regard the three-qubit states as real versions of the usual qubits called rebits [71]. In this case the SLOCC group should be modified accordingly to three copies of $GL(2, \mathbb{R})$. Restricting to determinant one transformations what we get is precisely the symmetry group of the STU model at the classical level, ie. $SL(2, \mathbb{R})^{\otimes 3}$. After implementing the (quantum) constraint coming from the usual Dirac-Zwanziger charge quantization the group we get is the U-duality group of the model namely $SL(2, \mathbb{Z})^{\otimes 3}$.

Now let us have a look at the constraint $D(\psi) < 0$. It can be shown that we can relax this constraint as well, provided we are willing to embark in the rich field of non-BPS black hole solutions [72, 73] for which $D(\psi) > 0$. Moreover, as discussed in the paper of Kallosh and Linde [2] there are also solutions for which $D(\psi) = 0$. These are called small black holes, referring to the fact that though they have vanishing Bekenstein-Hawking entropy, they can develop a nonvanishing entropy via higher-order and quantum corrections [74]. The final result of these considerations is that the classification of entanglement types of three rebits under the SLOCC group $[SL(2, \mathbb{R})]^{\otimes 3}$ can be mapped to the classification of different types of black holes solutions in the STU model [2, 3, 59] and vice versa.

In summary: the formula for the macroscopic black hole entropy in the STU model can be expressed in terms of the three-tangle which is a triality and U-duality invariant tripartite measure of entanglement as

$$S = \frac{\pi}{2} \sqrt{\tau_{ABC}(\psi)} = \pi \sqrt{|D(\psi)|} \quad (2.9)$$

where $|\psi\rangle$ is an unnormalized three-rebit state with amplitudes being the eight quantized charges. Cayley's hyperdeterminant is negative for BPS, and positive for non-BPS large black holes. These have nonzero horizon area and nonzero semiclassical Bekenstein-Hawking entropy. For small black holes we have $D(\psi) = 0$.

The question left to be answered is whether such rebits can somehow be embedded into the realm of genuine *complex* three-qubit states. Within the framework of conventional quantum information theory this problem has already been discussed [75]. In order to do this also in our black hole context we clearly have to see how other ingredients of the STU model (namely the complex scalar fields S , T and U) can be incorporated into the formalism. The introduction of such structures will be discussed later.

3. E_7 AND THE TRIPARTITE ENTANGLEMENT OF SEVEN QUBITS

3.1. Embedding the STU model

Having discussed the STU black hole entropy and its connection to three-qubit entanglement, the question now is whether we can extend our considerations to more general charge configurations. The extremal spherically symmetric black hole solutions in $\mathcal{N} = 8$ supergravity [8, 9] are defined by 28 + 28 electric/magnetic charges and the entropy formula is given by the square root of the quartic Cartan-Cremmer-Julia $E_{7(7)}$ invariant [9, 76, 77]

$$S = \pi \sqrt{|I_4|} \quad (3.1)$$

where the Cartan form

$$I_4 = -\text{Tr}(xy)^2 + \frac{1}{4}(\text{Tr}xy)^2 - 4(\text{Pf}x + \text{Pf}y) \quad (3.2)$$

depends on the 8×8 antisymmetric quantized charge matrices x and y . The 28 independent components of x and y correspond to electric and magnetic charges, respectively. From an M-theoretic point of view these charges originate from wrapping configurations of membranes on the extra dimensions, as discussed in section 6.

An alternative (the Cremmer-Julia) form of this invariant is given in terms of the 8×8 complex central charge matrix \mathcal{Z}

$$I_4 = \text{Tr}(\mathcal{Z}\bar{\mathcal{Z}})^2 - \frac{1}{4}(\text{Tr}\mathcal{Z}\bar{\mathcal{Z}})^2 + 4(\text{Pf}\mathcal{Z} + \text{Pf}\bar{\mathcal{Z}}) \quad (3.3)$$

where the overbars refer to complex conjugation. The definition of the Pfaffian is

$$\text{Pf}\mathcal{Z} = \frac{1}{2^4 \cdot 4!} \epsilon^{ABCDEFGH} \mathcal{Z}_{AB} \mathcal{Z}_{CD} \mathcal{Z}_{EF} \mathcal{Z}_{GH}. \quad (3.4)$$

The relation between the Cremmer-Julia and Cartan forms can be established by using the relation

$$\mathcal{Z}_{AB} = -\frac{1}{4\sqrt{2}}(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB} \quad (3.5)$$

where summation through the indices A, B is implied only for $A < B$. Here $(\Gamma^{IJ})_{AB}$ are the generators of the $SO(8)$ algebra, where (IJ) are the vector indices ($I, J = 0, 1, \dots, 7$) and (AB) are the spinor ones ($A, B = 0, 1, \dots, 7$). Triality of $SO(8)$ ensures that we can transform between its vector and spinor representations. A consequence of this is that we can also invert the relation of (3.5) and express $x^{IJ} + iy_{IJ}$ in terms of the central charge matrix \mathcal{Z}_{AB} .

Let

$$\begin{aligned} x^{01} + iy_{01} &= -\psi_7 - i\psi_0, & x^{34} + iy_{34} &= \psi_1 + i\psi_6 \\ x^{26} + iy_{26} &= \psi_2 + i\psi_5, & x^{57} + iy_{57} &= \psi_4 + i\psi_3 \end{aligned} \quad (3.6)$$

where the remaining components of x and y are set to zero. Then a calculation shows that

$$I_4 = -D(\psi) \quad (3.7)$$

where $D(\psi)$ is Cayley's hyperdeterminant of (2.3). This result suggests that we should be able to obtain the three-qubit interpretation of the STU model as a consistent truncation of a larger entangled system living within our $\mathcal{N} = 8$, $D = 4$ supergravity theory.

By a transformation of the form $\mathcal{Z} \mapsto U^t \mathcal{Z} U$, where $U \in SU(8)$, \mathcal{Z} can be brought to the form

$$\mathcal{Z}_{canonical} = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \otimes \varepsilon, \quad (3.8)$$

where ε is the antisymmetric 2×2 matrix $\varepsilon^{12} = 1$ and all four z_j can be chosen to have the same phase, or three of the z_i can be chosen to be real [78]. Our choice of (3.6) can then be related to this canonical form as

$$z_1 = \frac{1}{\sqrt{8}}(-\psi_7 + \psi_1 + \psi_2 + \psi_4 + i(-\psi_0 + \psi_6 + \psi_5 + \psi_3)), \quad z_2 = \frac{1}{\sqrt{8}}(-\psi_7 - \psi_1 + \psi_2 - \psi_4 + i(-\psi_0 - \psi_6 + \psi_5 - \psi_3))$$

where z_3 and z_4 is obtained from z_2 by a cyclic permutation of the $+$ sign. As a result of these considerations it can be shown that the STU truncation is a natural one related to the canonical form of the central charge matrix [79]. Can we interpret this truncation as a one arising from some larger entangled system?

3.2. E_7 in the cyclic representation

Our success with the three-qubit interpretation of the STU model is clearly related to the underlying $[SL(2, \mathbb{R})]^{\otimes 3}$ symmetry group of the corresponding $\mathcal{N} = 2$ supergravity which can be related to *real states* or *rebits* which also transform as the $(2, 2, 2)$ of the complex SLOCC subgroup $[SL(2, \mathbb{C})]^{\otimes 3}$ of a three-qubit system. However, in the $\mathcal{N} = 8$ context the symmetry group in question is $E_{7(7)}$ which is not of the product form hence a qubit interpretation seems to be impossible. However, we know that the 56 charges of the $\mathcal{N} = 8$ model transform as the fundamental 56-dimensional representation of $E_{7(7)}$. We can try to arrange these 56 charges as the integer-valued amplitudes of a reference state. However, 56 is not a power of 2 so the entanglement of this reference state if it exists at all should be of unusual kind. A trivial observation is that $56 = 7 \times 8$ hence the direct sum of seven copies of three-qubit state spaces produces the right count. Moreover, a multiqubit description is possible if the complexification $E_7(\mathbb{C})$ contains the product of some number of copies of the SLOCC subgroup $SL(2, \mathbb{C})$. Since the rank of E_7 is seven we expect that it should contain seven copies of $SL(2, \mathbb{C})$ groups. Hence this 56-dimensional representation space might be constructed as some combination of tripartite states of seven qubits. This construction is indeed possible [15, 16]. The relevant decomposition of the **56** of $E_7(\mathbb{C})$ with respect to the $[SL(2, \mathbb{C})]^{\otimes 7}$ subgroup is [15]

$$\begin{aligned} \mathbf{56} \rightarrow & (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}) + (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}). \end{aligned} \quad (3.9)$$

While this is clearly not a subspace of the 7-qubit Hilbert space, it is in fact a subspace of seven qutrits closed under $SL(2, \mathbb{C})^{\otimes 7} \subset SL(3, \mathbb{C})^{\otimes 7}$ [15], and so admits a conventional interpretation despite the appearance of the direct sum. Let us now formally

replace the 2's with 1's, and the 1's with 0's, and form a 7×7 matrix by regarding the seven vectors obtained in this way as its rows. Let the rows correspond to lines and the columns to points, and the location of a "1" in the corresponding slot correspond to incidence. Then this correspondence results in the incidence matrix of the Fano plane in the cyclic, or Paley [55], realization. Changing the roles of rows and columns we obtain the incidence structure of the dual Fano plane. Hence the multiqubit state we are searching for is a state associated with the incidence geometry of the Fano plane (see Figure (1)).

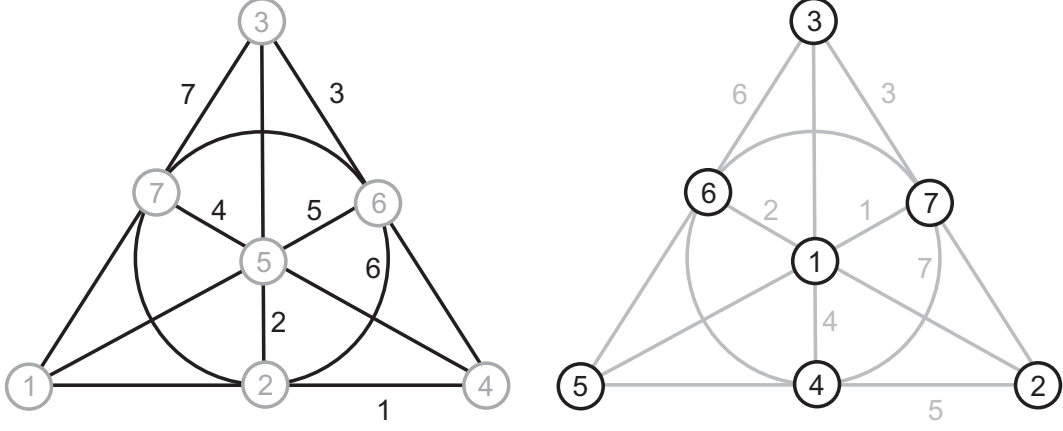


FIG. 1: The Fano plane and its dual.

Let us reproduce here this incidence matrix with the following labelling for the rows (r) and columns (c)

$$\begin{pmatrix} r/c & A & B & C & D & E & F & G \\ a & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ d & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ f & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ g & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a_{ABD} \\ b_{BCE} \\ c_{CDF} \\ d_{DEG} \\ e_{EFA} \\ f_{FGB} \\ g_{GAC} \end{pmatrix} \quad (3.10)$$

where we also displayed the important fact that this labelling automatically defines the index structure for the amplitudes of seven three-qubit states formed out of seven qubits A, B, C, D, E, F, G (Alice, Bob, Charlie, Daisy, Emma, Fred and George). If we introduce the notation $V_{ijk} \equiv V_i \otimes V_j \otimes V_k$ where $i, j, k \in \{A, B, C, D, E, F, G\}$ then the **56** of E_7 denoted by \mathcal{H} decomposes as

$$\mathcal{H} = V_{ABD} \oplus V_{BCE} \oplus V_{CDF} \oplus V_{DEG} \oplus V_{EFA} \oplus V_{FGB} \oplus V_{GAC}. \quad (3.11)$$

Clearly this structure encompasses an unusual type of entanglement; entanglement is usually associated with tensor products, however here we also encounter *direct sums*. One can regard the seven tripartite sectors as seven superselection sectors corresponding to seven different *STU* truncations [15, 16]. This structure is usually referred to in the literature as the tripartite entanglement of seven qubits. When the amplitudes are reinterpreted as quantized charges the elements of \mathcal{H} are states associated to the incidence geometry of the Fano plane. In order to understand which amplitudes of the seven three-qubit states correspond to electric, and magnetic charges we need to relate the amplitudes of the correspondence of (3.10) to the matrices x^{IJ} and y_{IJ} of the Cartan form of (3.2). Using the decimal labelling we obtain the so-called Cartan-Fano dictionary [59].

$$x^{IJ} = \begin{pmatrix} 0 & -a_7 & -b_7 & -c_7 & -d_7 & -e_7 & -f_7 & -g_7 \\ a_7 & 0 & f_1 & d_4 & -c_2 & g_2 & -b_4 & -e_1 \\ b_7 & -f_1 & 0 & g_1 & e_4 & -d_2 & a_2 & -c_4 \\ c_7 & -d_4 & -g_1 & 0 & a_1 & f_4 & -e_2 & b_2 \\ d_7 & c_2 & -e_4 & -a_1 & 0 & b_1 & g_4 & -f_2 \\ e_7 & -g_2 & d_2 & -f_4 & -b_1 & 0 & c_1 & a_4 \\ f_7 & b_4 & -a_2 & e_2 & -g_4 & -c_1 & 0 & d_1 \\ g_7 & e_1 & c_4 & -b_2 & f_2 & -a_4 & -d_1 & 0 \end{pmatrix} \quad (3.12)$$

$$y^{IJ} = \begin{pmatrix} 0 & -a_0 & -b_0 & -c_0 & -d_0 & -e_0 & -f_0 & -g_0 \\ a_0 & 0 & f_6 & d_3 & -c_5 & g_5 & -b_3 & -e_6 \\ b_0 & -f_6 & 0 & g_6 & e_3 & -d_5 & a_5 & -c_3 \\ c_0 & -d_3 & -g_6 & 0 & a_6 & f_3 & -e_5 & b_5 \\ d_0 & c_5 & -e_3 & -a_6 & 0 & b_6 & g_3 & -f_5 \\ e_0 & -g_5 & d_5 & -f_3 & -b_6 & 0 & c_6 & a_3 \\ f_0 & b_3 & -a_5 & e_5 & -g_3 & -c_6 & 0 & d_6 \\ g_0 & e_6 & c_3 & -b_5 & f_5 & -a_3 & -d_6 & 0 \end{pmatrix}. \quad (3.13)$$

As explained elsewhere the structure of these matrices is encoded into the structure constants of the dual Fano plane (see the second of Figure 1) and the structure constants of the octonions [59]. One can now see that the choice of (3.6) corresponds to identification of $|\psi\rangle$ with the three-qubit state $|a\rangle$ with amplitudes a_{ABD} , $A, B, D = 0, 1$ built from the qubits of Alice, Bob and Daisy, and the remaining 48 amplitudes are zero. This STU truncation is just one of seven possibilities corresponding to the three-qubit states $|b\rangle, \dots |g\rangle$. The quartic invariant I_4 truncates to $-D(a), \dots -D(g)$ in these seven possible cases. This relates the black hole entropies of the seven possible STU sectors to the corresponding three-tangles of the relevant charge states.

Concerning the seven possible STU truncations it is important to realise that there is an automorphism α of order seven which transforms cyclically the amplitudes $a, b, \dots g$ of the relevant three qubit states into each other. Note, α transforms cyclically the points $1, 2, \dots 7$ of the dual Fano plane of Figure 1. One can find an 8×8 orthogonal matrix representation $\mathcal{D}(\alpha)$ acting on the central charge as $\mathcal{Z} \mapsto \mathcal{D}(\alpha)\mathcal{Z}\mathcal{D}^T(\alpha)$. It can be expressed [18] in terms of the “controlled not” (CNOT) operators [4] as

$$\mathcal{D}(\alpha) = (C_{12}C_{21})(C_{12}C_{31})C_{23}(C_{12}C_{31}). \quad (3.14)$$

It can be shown [18] that using \mathcal{Z} as given by (3.5) with a convenient representation for the gamma matrices the effect of $\mathcal{D}(\alpha)$ is to rotate the seven groups of three-qubit amplitudes showing up in (3.12)-(3.13) cyclically.

This representation for the automorphism of order seven can be generalized [18, 59] to one for the full automorphism group of the Fano plane which is $PSL_2(7)$. Moreover, it turns out that $PSL_2(7)$ can also be represented on the 28 charges regarded as composites of electric and magnetic ones with their incidence geometry corresponding to the Coxeter graph [18]. This configuration shows up as a subgeometry of an object called the split-Cayley hexagon related to the incidence geometry of the real three-qubit Pauli group [18, 57]. Now $PSL_2(7)$ can be embedded into the Weyl group $W(E_7)$, which is a subgroup of the full U-duality group $E_7(\mathbb{Z})$ implementing electric-magnetic duality [80]. The fact that the Weyl-group of E_7 is naturally connected to three-qubit quantum gates was first emphasized by Planat and Kibler [81]. For a recent elaboration on this connection with a description of $W(E_7)$ and three qubits in terms of symplectic transvections [57] see the paper of Cherchiai and van Geemen [58].

3.3. E_7 and the Hamming code

Let us now see yet another realization of the tripartite entanglement of seven qubits living inside E_7 . This realization is related to a famous error correcting code: the Hamming code. As a starting point let us consider the matrix of the three-qubit discrete Fourier transformation i.e. the tensor product $H \otimes H \otimes H$ of three Hadamard gates where

$$H \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (3.15)$$

Delete now the first column of the matrix $H \otimes H \otimes H$ and replace the -1 s with 0 s in the remaining 8×7 matrix. Alternatively we can replace the $+1$ s with 0 s and the -1 s with 1 s. Then we obtain the following matrices which are complements of each other

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \quad (3.16)$$

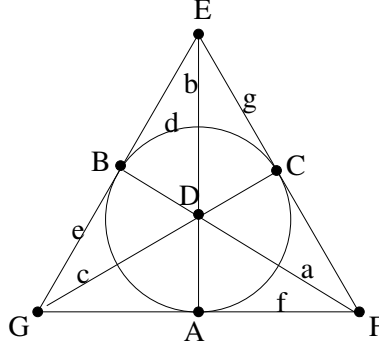


FIG. 2: The Hadamard labelling convention for the points and lines of the Fano plane.

One can then regard the rows of these matrices as seven binary digit codewords encoding messages of *four* digits. For this purpose let us now regard the *first*, *second* and *fourth* digits as *check digits*. The remaining ones are the *message digits*. Hence for example the codeword $(0, 1, 0, 1, 0, 1, 0)$ encodes the message 0010 and the check digits are 011. If we would like to send four message bits through a noisy channel we can encode our 16 possible 4 digit message bits into our 16 seven digit long codewords as discussed above. Let us suppose that the noisy channel has the effect of flipping just one of the seven bits. The recipient would like to know whether the seven bit sequence has been corrupted or not. Moreover, if it is corrupted she would like to correct it unambiguously. In order to see that she can perform this task just notice that all of our codewords differ from each other in *at least three digits*. If we define the Hamming distance between two codewords as the number of places in which the codewords differ we see that all pairs of our codewords have distance at least three. Now if one error is made in the transmission then the received binary sequence will still be closer to the original one than to any other. As a result the received sequence can be unambiguously corrected by choosing the codeword from the list which is the closest to it.

Now our aim is to demonstrate that the two matrices of (3.16) related to the codewords of the Hamming code encode another version of the tripartite entanglement of seven qubits and the structure of the Lie-algebra of E_7 . Let us first use the first of the two matrices of (3.16) as the incidence matrix of yet another copy of the Fano plane in the Hadamard parametrization [55]. For this purpose write the incidence matrix with the following labelling for the rows (r) and columns (c)

$$\begin{pmatrix} r/c & A & B & C & D & E & F & G \\ a & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ b & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ f & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ g & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{BDF} \\ b_{ADE} \\ c_{CDG} \\ d_{ABC} \\ e_{BEG} \\ f_{AFG} \\ g_{CEF} \end{pmatrix} \quad (3.17)$$

where this labelling automatically defines the index structure for the amplitudes of seven three-qubit states formed out of seven qubits A, B, C, D, E, F, G . This convention also fixes the labelling of lines and points of the Fano plane, see Figure 2.

To the points again we associate qubits and to the lines three-qubit systems with vector spaces $V_{BDF}, V_{ADE}, \dots, V_{CEF}$. A list of these three-qubit Hilbert spaces \mathcal{H}_σ , $\sigma \in \mathbb{Z}_2^3 - (000)$ is given by the correspondence

$$(\mathcal{H}_{001} \ \mathcal{H}_{010} \ \mathcal{H}_{011} \ \mathcal{H}_{100} \ \mathcal{H}_{101} \ \mathcal{H}_{110} \ \mathcal{H}_{111}) \leftrightarrow (V_{BDF} \ V_{ADE} \ V_{CDG} \ V_{ABC} \ V_{BEG} \ V_{AFG} \ V_{CEF}).$$

Now we switch to a new ordering of the spaces \mathcal{H}_σ according to the rule $(1, 2, 3, 4, 5, 6, 7) \rightarrow ((100), (010), (110), (001), (101), (011), (111))$ which is the reverse binary labelling. This yields our definition for the representation space of the **56** of E_7 in terms of the spaces \mathcal{H}_σ as

$$\mathcal{H} = V_{ABC} \oplus V_{ADE} \oplus V_{AFG} \oplus V_{BDF} \oplus V_{BEG} \oplus V_{CDG} \oplus V_{CEF}. \quad (3.18)$$

In order to shed some light on the possibility of describing also the structure of the Lie-algebra of E_7 in terms of data provided by the Hamming code let us consider the second matrix of (3.16).

The Lie-algebra of E_7 has 133 dimensions. We clearly have $\mathfrak{sl}(2)^{\oplus 7}$ as a subalgebra of dimension $7 \times 3 = 21$. These 21 generators act on \mathcal{H} of (3.18) via the well known action of the SLOCC subgroup. To define the remaining 112 generators and their action on \mathcal{H} we consider the complements of the lines of the Fano plane of Figure 2. These seven sets of four points form seven *quadrangles*. Since we have already attached to the points of the Fano plane qubits, and this assignment automatically

defined our three-qubit states corresponding to the lines, it then follows that the quadrangles define seven four-qubit states. They form the 112 dimensional complex vector space

$$W \equiv V_{DEFG} \oplus V_{BCFG} \oplus V_{BCDE} \oplus V_{ACEG} \oplus V_{ACDF} \oplus V_{ABEF} \oplus V_{ABDG} \quad (3.19)$$

which we can use as the space of E_7 generators not belonging to the SLOCC subalgebra. Notice that since the complements of the quadrangles are lines that can be associated to seven three-qubit states one can label each of these 16 dimensional spaces as $\mathcal{W}_{001}, \mathcal{W}_{010}, \dots, \mathcal{W}_{111}$.

Let us denote the basis vectors of the corresponding four-qubit spaces in the computational base as $(T_{ACEG}, \dots, T_{ABDG})$. A clear indication that we are on the right track for defining the \mathfrak{e}_7 algebra via four-qubit states comes from the possibility of defining the Lie-bracket on W using,

$$[T_{ACEG}, T_{BC'FG'}] = \Phi(ACEG, BC'FG') \varepsilon_{CC'} \varepsilon_{GG'} T_{ABEF} \quad (3.20)$$

where in this example the pair CG is common to both quadrangles, $ACEG$ and $BCFG$. It can be shown that the explicit form for Φ is arising from the octonionic multiplication rule that is in turn also encoded into the Hamming code via the structure of the Fano plane. Adding also the 21 generators of the SLOCC group as an extra vector space \mathcal{W}_{000} one can show that after introducing the 133 dimensional vector space $\mathcal{W} \equiv \mathcal{W}_{000} \oplus W$ the Lie bracket can be extended using the obvious commutators. Denoting this extended bracket by $[\cdot, \cdot]$ one can show [55, 82, 83]

$$e_7 = (\mathcal{W}, [\cdot, \cdot]). \quad (3.21)$$

As we see \mathcal{W} has a deep connection with the division algebra of octonions. In technical terms \mathfrak{e}_7 , as a vector space, has an octonionic grading [82].

Using this formalism based on the Hamming code one can show that the generators of \mathfrak{e}_7 can be written as combinations of tripartite entanglement transformations [16, 55]. Some of them are of SLOCC form (those operating in the diagonal blocks), while others generate correlations between the different tripartite sectors. One can also show that the representation theoretic details are entirely encoded in a so-called $(7, 3, 1)$ design and its complementary $(7, 4, 2)$ one [55], which correspond to the two matrices of (3.16) and are related to lines and quadrangles of the smallest finite projective plane: the Fano plane. Moreover, these designs are described in a unified form via the nontrivial codewords of the Hamming code of (3.16). The Hamming code in turn is clearly related to the Hadamard matrix (3.15) which is the discrete Fourier transform on three-qubits. We will see in later sections that such Hadamard transformations on three-qubits also play a role in obtaining a nice characterisation of BPS and non-BPS solutions of the STU truncation. This suggests that black hole solutions of more general type might be understood in a framework related to error correcting codes.

3.4. The structure of the E_7 symmetric black hole entropy formula

We have already discussed Cartan's quartic invariant (3.2) well-known from studies concerning $SO(8)$ supergravity [9, 76, 77]. I_4 is the singlet in the tensor product representation $\mathbf{56} \times \mathbf{56} \times \mathbf{56} \times \mathbf{56}$. Its explicit form in connection with stringy black holes with their $E_{7(7)}$ symmetric area form [77] is given either in the Cremmer-Julia form [9] in terms of the complex 8×8 central charge matrix Z or in the Cartan form [76] in terms of two real 8×8 ones x and y containing the quantized electric and magnetic charges of the black hole. Let us now present its new form in terms of the 56 amplitudes of our seven qubits [15]. In the Hadamard representation of (3.18) its new expression is

$$\begin{aligned} I_4 = & \frac{1}{2}(a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4) + \\ & 2[a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + e^2f^2 + f^2g^2 + g^2a^2 + \\ & a^2c^2 + b^2d^2 + c^2e^2 + d^2f^2 + e^2g^2 + f^2a^2 + g^2b^2 + \\ & a^2d^2 + b^2e^2 + c^2f^2 + d^2g^2 + e^2a^2 + f^2b^2 + g^2c^2] \\ & + 8[aceg + bcfg + abef + defg + acdf + bcde + abdg]. \end{aligned} \quad (3.22)$$

Here we have for example

$$bcde = \varepsilon^{A_1 A_3} \varepsilon^{B_3 B_4} \varepsilon^{C_2 C_3} \varepsilon^{D_1 D_2} \varepsilon^{E_1 E_4} \varepsilon^{G_2 G_4} b_{A_1 D_1 E_1} c_{C_2 D_2 G_2} d_{A_3 B_3 C_3} e_{B_4 E_4 G_4}.$$

The remaining terms can be described in a unified manner by employing the following definition. Let us consider for example the two three-qubit states $|d\rangle$ and $|b\rangle$ with amplitudes d_{ABC} and b_{ADE} made of five different qubits A, B, C, D, E with qubit A as the common one. For this situation we define

$$d^2 b^2 \equiv \mathcal{Q}(d, b) = \varepsilon^{A_1 A_3} \varepsilon^{B_1 B_2} \varepsilon^{C_1 C_2} \varepsilon^{A_2 A_4} \varepsilon^{D_3 D_4} \varepsilon^{E_3 E_4} d_{A_1 B_1 C_1} d_{A_2 B_2 C_2} b_{A_3 D_3 E_3} b_{A_4 D_4 E_4}.$$

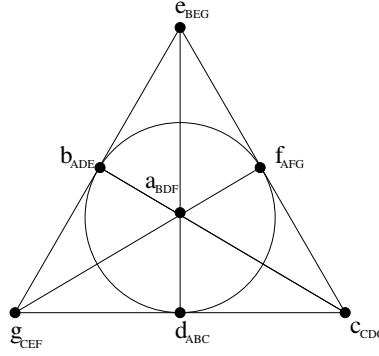


FIG. 3: The dual Fano plane. To its points now we attached three qubit states with the representative amplitudes indicated. To the lines we associate the common qubits these tripartite states share.

In this notation

$$d^4 \equiv \mathcal{Q}(d, d) = -2D(d)$$

i.e. -2 times the usual expression for Cayley's hyperdeterminant.

Another important observation is that the terms occurring in (3.22) can be understood using the *dual* Fano plane. To see this note that the Fano plane is a projective plane hence we can use projective duality to exchange the role of lines and planes. Originally we attached qubits to the points, and tripartite systems to the lines of the Fano plane. Now we take the dual perspective, and attach the tripartite states to the points and qubits to the lines of the dual Fano plane see Figure 3. In the ordinary Fano plane the fact that three lines intersect in a unique point corresponded to the fact that any three entangled tripartite systems share a unique qubit. In the dual perspective this entanglement property corresponds to the geometric one that three points are always lying on a unique line. For example let us consider the three points corresponding to the tripartite states with amplitudes d , b , and f . Looking at Figure 3 these amplitudes define the corresponding points lying on the line dbf . This line is defined by the common qubit these tripartite states share namely qubit A .

In the dual Fano plane we have seven points, with seven tripartite states attached to them. The corresponding entanglement measures are proportional to seven copies of Cayley's hyperdeterminant, then in I_4 we have the terms $a^4, b^4, c^4, d^4, e^4, f^4$ and g^4 . We also have seven lines with three tripartite states on each of them. We can group the 21 terms of the form a^2b^2 etc. into seven groups associated to such lines. They describe the pairwise entanglement between the three different tripartite systems (sharing a common qubit). For example for the line dbf we have the terms b^2d^2 , d^2f^2 and b^2f^2 describing such pairwise entanglements. Finally we have seven quadrangles (as complements to the lines) with four entangled tripartite systems giving rise to the last seven terms in I_4 .

Apart from immediately identifying the seven different STU truncations, the form of (3.22) has many other virtues. One of them is that we can easily understand some of the nontrivial truncations and as an extra bonus we can also quickly realize their finite geometric meaning. As an example let us consider the decomposition of the **56** of E_7

$$\mathbf{56} \rightarrow (\mathbf{2}, \mathbf{12}) \oplus (\mathbf{1}, \mathbf{32}) \quad (3.23)$$

with respect to the maximal subgroup $SL(2, \mathbb{C}) \times SO(12, \mathbb{C})$. Notice that the $(\mathbf{2}, \mathbf{12})$ part of the representation space \mathcal{H} consists of the amplitudes of the form

$$\begin{pmatrix} d_{ABC} \\ b_{ADE} \\ f_{AFG} \end{pmatrix} \in \mathcal{H}_{(\mathbf{2}, \mathbf{12})} \equiv V_{ABC} \oplus V_{ADE} \oplus V_{AFG} = V_A \otimes (V_{BC} \oplus V_{DE} \oplus V_{FG}) \quad (3.24)$$

and the $(\mathbf{1}, \mathbf{32})$ part of the ones

$$\begin{pmatrix} a_{BDF} \\ e_{BEG} \\ c_{CDG} \\ g_{CEF} \end{pmatrix} \in \mathcal{H}_{(\mathbf{1}, \mathbf{32})} \equiv V_{BDF} \oplus V_{BEG} \oplus V_{CDG} \oplus V_{CEF}. \quad (3.25)$$

We see that the $(\mathbf{2}, \mathbf{12})$ space consists of all the amplitudes sharing qubit A in common, and the $(\mathbf{1}, \mathbf{32})$ all those *excluding* qubit A .

It is also clear that by writing our representation space as

$$V_{ADE} \oplus V_A \otimes (V_{BC} \oplus V_{FG}) \oplus V_D \otimes (V_{BF} \oplus V_{CG}) \oplus V_E \otimes (V_{BG} \oplus V_{CF}). \quad (3.26)$$

one can easily understand the decomposition

$$(\mathbf{2}, \mathbf{12}) \oplus (\mathbf{1}, \mathbf{32}) \rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{8}_v) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{8}_s) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{8}_s) \quad (3.27)$$

with respect to the inclusion $\text{SL}(2) \times \text{SL}(2) \times \text{SL}(2) \times \text{SO}(4, 4) \subset \text{SL}(2) \times \text{SO}(6, 6)$.

Let us discuss the meaning of the $(\mathbf{2}, \mathbf{12})$ truncation in the black hole context. In this case the corresponding groups are real, hence in the supergravity approximation we have $\text{SL}(2, \mathbb{R}) \times \text{SO}(6, 6)$. We have in this case 2×12 charges. In the quantum theory they are integer valued and the U-duality group is broken to $\text{SL}(2, \mathbb{Z}) \times \text{SO}(6, 6, \mathbb{Z})$. The groups $\text{SL}(2, \mathbb{Z})$ and $\text{SO}(6, 6, \mathbb{Z})$ are the S and T duality transformations of toroidally compactified type IIA string theory [11]. For the black hole solutions in the corresponding models one can obtain entropy formulae that are truncations of (3.22) with integer amplitudes. Since we have seven lines in the dual Fano plane such truncations can be obtained in seven different ways. As a particular truncation one can take for example the line dbf in the dual Fano plane. The relevant truncation of I_4 interpreted as a measure of pure state entanglement one can take

$$\tau_3^{(3)} = 2|b^4 + d^4 + f^4 + 2(b^2 d^2 + d^2 f^2 + b^2 f^2)| \quad (3.28)$$

where the notation $\tau_3^{(3)}$ indicates that now we have three tripartite states. Now we write the state corresponding to the line dbf in the form

$$|\psi\rangle = \sum_{ABCDEF=0,1} |A\rangle \otimes (d_{ABC}|BC\rangle + b_{ADE}|DE\rangle + f_{AFG}|FG\rangle). \quad (3.29)$$

This notation clearly displays that qubit A is entangled with the remaining pairs $(BC)(DE)(FG)$. Recalling that this state transforms as the $(\mathbf{2}, \mathbf{12})$ of $\text{SL}(2) \times \text{SO}(6, 6)$ we can write,

$$|\psi\rangle = \sum_{A\mu} \psi_{A\mu} |A\rangle \otimes |\mu\rangle, \quad A = 0, 1, \quad \mu = 1, 2, \dots, 12. \quad (3.30)$$

Introducing the notation

$$p^\mu \equiv \psi_{0\mu} = \begin{pmatrix} d_{0BC} \\ b_{0DE} \\ f_{0FG} \end{pmatrix}, \quad q^\mu \equiv \psi_{1\mu} = \begin{pmatrix} d_{1BC} \\ b_{1DE} \\ f_{1FG} \end{pmatrix}. \quad (3.31)$$

In these variables for $\tau_3^{(3)}$ we get the following expression

$$\tau_3^{(3)} = 4|(\mathbf{p}\mathbf{p})(\mathbf{q}\mathbf{q}) - (\mathbf{p}\mathbf{q})^2| \quad (3.32)$$

where the scalar products above are defined with respect to the 12×12 block-diagonal matrix containing three copies of $\varepsilon \otimes \varepsilon$. Now in the black hole analogy p^μ and q^μ are integer and the measure of entanglement in (3.28) can be related to the black hole entropy [84]

$$S = \frac{\pi}{2} \sqrt{\tau_3^{(3)}} \quad (3.33)$$

coming from the truncation of the $\mathcal{N} = 8$ theory with $E_{7(7)}(\mathbb{Z})$ symmetry to $\mathcal{N} = 4$ supergravity coupled to 6 vector multiplets with $\text{SL}(2, \mathbb{Z}) \times \text{SO}(6, 6, \mathbb{Z})$ U-duality. From the string theoretical point of view this sector describes the NS-NS charges. Note, that in the cyclic representation of (3.11) the formula above can also be reinterpreted as Cayley's hyperdeterminant over the imaginary quaternions [59]. For a similar discussion of the $(\mathbf{1}, \mathbf{32})$ truncation of I_4 and its interpretation as an entanglement measure featuring the 32 so called R-R charges we refer the reader to the literature [59].

4. E_6 AND THE BIPARTITE ENTANGLEMENT OF THREE QUTRITS

4.1. The Octonions and the cubic invariant

In quantum information one can consider entangled states representing quantum systems with more than two states (qubits). Apart from qubits the simplest objects to consider are entangled three-state systems called *qutrits*. A qutrit is an element of

\mathbb{C}^3 and the corresponding SLOCC group acting on it is $GL(3, \mathbb{C})$. The entanglement measures for such systems should come from relative invariants under the action of the multipartite local SLOCC group. For example as the simplest relative SLOCC invariant for a two-qutrit state of the form

$$|\psi\rangle = \sum_{A,B=0,1,2} \psi_{AB} |A\rangle \otimes |B\rangle \quad (4.1)$$

one can take the determinant $\text{Det}(\psi)$ of the 3×3 matrix ψ_{AB} . This quantity is obviously a relative invariant which is invariant under the $SL(3, \mathbb{C}) \times SL(3, \mathbb{C})$ subgroup of two-qutrit SLOCC transformations. The classification of two-qutrit states is very simple. Different SLOCC classes are labelled by the rank of the 3×3 matrix ψ_{AB} .

Based on our experience of relating special entangled systems built from few *real qubits* (or rebits) to the structure of black hole entropy formulae for $D = 4$ supergravity theories with $\mathcal{N} = 8, 4, 2$ supersymmetries the question now is the following: can we find entangled systems of real *qutrits* that can be related to black hole entropy formulae of other kind? This generalization is indeed possible, provided we consider black hole, and black string solutions in five dimensions [17]. Moreover, we can then also use these structures in the complex domain, as new entanglement measures of some hypothetical entangled system.

In order to see how these structures arise let us recall that magic $\mathcal{N} = 2, D = 5$ supergravities [37–39, 59] coupled to 5, 8, 14 and 26 vector multiplets with symmetries $SL(3, \mathbb{R}), SL(3, \mathbb{C}), SU^*(6)$ and $E_{6(-26)}$ can be described by Jordan algebras of 3×3 Hermitian matrices with entries taken from the reals, complexes, quaternions and octonions. It is also known that in these cases we have black hole solutions that have *cubic invariants* whose square roots yield the corresponding black hole entropy [85]. Moreover, we can also replace in these Jordan algebras the division algebras by their split versions. For example, in the case of split octonions we arrive at the $\mathcal{N} = 8, D = 5$ supergravity [86] with 27 Abelian gauge fields transforming in the fundamental of $E_{6(6)}$. In this theory the corresponding black hole solutions have an $E_{6(6)}(\mathbb{Z})$ symmetric entropy formula [59, 87, 88]. It is also important to note that the magic $\mathcal{N} = 2$ supergravities associated with the reals, complexes and quaternions can be obtained as consistent reductions of the $\mathcal{N} = 8$ theory [85] based on the split octonions. On the other hand, the $\mathcal{N} = 2$ supergravity based on the division algebra of the octonions is exceptional since it cannot be obtained from the split octonionic $\mathcal{N} = 8$ theory by truncation.

Since in all these cases the black hole entropy is given in terms of a cubic invariant, to relate them to entangled systems of some kind we need to first understand the structure of these invariants.

An element of a magic cubic Jordan algebra can be represented as a 3×3 Hermitian matrix with entries taken from a division algebra \mathbb{A} , i. e. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} . Explicitly, we have

$$J_3(Q) = \begin{pmatrix} q_1 & Q^v & \overline{Q^s} \\ \overline{Q^v} & q_2 & Q^c \\ Q^s & \overline{Q^c} & q_3 \end{pmatrix} \quad q_i \in \mathbb{R}, \quad Q^{v,s,c} \in \mathbb{A} \quad (4.2)$$

where an overbar refers to conjugation in \mathbb{A} . These charge configurations describe electric black holes of the $\mathcal{N} = 2, D = 5$ magic supergravities [37–39, 59]. In the octonionic case the superscripts of Q refer to the fact that the fundamental 27-dimensional representation of the U-duality group $E_{6(-26)}$ decomposes under the subgroup $SO(8)$ to three 8-dimensional representations (vector, spinor and conjugate spinor connected by triality) plus three singlets corresponding to the $q_i, i = 1, 2, 3$. A general element in this case is of the form $Q = Q_0 + Q_1 e_1 + \dots + Q_7 e_7$, where the imaginary units e_1, e_2, \dots, e_7 satisfy the rules of the octonionic multiplication table. The norm of an octonion is $Q\overline{Q} = (Q_0)^2 + \dots + (Q_7)^2$. The real part of an octonion is defined as $\text{Re}(Q) = \frac{1}{2}(Q + \overline{Q})$. The magnetic analogue of $J_3(Q)$ is obtained by replacing Q with P referring now to the magnetic charges. $J_3(P)$ describes black strings related to the previous case by electric-magnetic duality.

The black hole entropy is given by the cubic invariant

$$I_3(Q) = q_1 q_2 q_3 - (q_1 Q^c \overline{Q^c} + q_2 Q^s \overline{Q^s} + q_3 Q^v \overline{Q^v}) + 2\text{Re}(Q^v Q^s Q^c) \quad (4.3)$$

as

$$S = \pi \sqrt{I_3(Q)} \quad (4.4)$$

and for the black string we get a similar formula with $I_3(Q)$ replaced by $I_3(P)$.

In the split octonionic case \mathbb{O}_s the norm is defined as

$$Q\overline{Q} = (Q_0)^2 + (Q_1)^2 + (Q_2)^2 + (Q_3)^2 - (Q_4)^2 - (Q_5)^2 - (Q_6)^2 - (Q_7)^2 \quad (4.5)$$

and the group preserving the cubic invariant of the corresponding Jordan algebra is $E_{6(6)}$, which decomposes similarly under $SO(4, 4)$. This is the case of $\mathcal{N} = 8$ supergravity with duality group $E_{6(6)}$ [89]. In the quantum theory the black hole/string charges become integer-valued and the relevant 3×3 matrices are defined over the *integral* octonions and *integral* split octonions, respectively. The U-duality groups are in this case broken to $E_{6(-26)}(\mathbb{Z})$ and $E_{6(6)}(\mathbb{Z})$. In all these cases the entropy formula is given by (4.3)–(4.4), with the norm given by either the usual one or its split analogue (4.5).

4.2. Qutrits and the cubic invariant

Since all the $\mathcal{N} = 2$ magic, aside from the octonionic case, supergravities can be obtained as consistent truncations of the $\mathcal{N} = 8$ split-octonionic case, let us consider the cubic invariant I_3 of (4.3) with the U-duality group $E_{6(6)}$. Let us also consider the decomposition of the 27-dimensional fundamental representation of $E_{6(6)}$ with respect to its $[\mathrm{SL}(3, \mathbb{R})]^{\otimes 3}$ subgroup,

$$E_{6(6)} \supset \mathrm{SL}(3, \mathbb{R})_A \times \mathrm{SL}(3, \mathbb{R})_B \times \mathrm{SL}(3, \mathbb{R})_C \quad (4.6)$$

under which

$$\mathbf{27} \rightarrow (\mathbf{3}', \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}', \mathbf{3}') \oplus (\mathbf{3}, \mathbf{1}, \mathbf{3}). \quad (4.7)$$

The above-given decomposition gives the bipartite entanglement of three-qutrits interpretation [17, 59] of the $\mathbf{27}$ of $E_{6(6)}$. Just as the tripartite entanglement of seven qubits was not a subspace of seven qubit, clearly this is not a subspace of the three qutrits. However, it is a subspace of three 7-dits closed under $[\mathrm{SL}(3, \mathbb{C})]^{\otimes 3} \subset [\mathrm{SL}(7, \mathbb{C})]^{\otimes 7}$ [17], and so once again admits a conventional interpretation despite the appearance of the direct sum. Neglecting the details, all we need is three 3×3 real matrices a, b and c with the index structure

$$a^A{}_B, \quad b^{BC}, \quad c_{CA}, \quad A, B, C = 0, 1, 2 \quad (4.8)$$

where the upper indices are transformed according to the (contragredient) $\mathbf{3}'$ and the lower ones by $\mathbf{3}$. The explicit dictionary between the qutrit amplitudes a, b and c above and the components of the Jordan algebra as given in (4.2) can be found in the literature [19, 59].

Now the new expression for the cubic invariant I_3 of (4.3) is [17]

$$I_3 = \mathrm{Det} J_3(Q) = a^3 + b^3 + c^3 + 6abc. \quad (4.9)$$

Here

$$a^3 = \frac{1}{6} \varepsilon_{A_1 A_2 A_3} \varepsilon^{B_1 B_2 B_3} a^{A_1}{}_{B_1} a^{A_2}{}_{B_2} a^{A_3}{}_{B_3}, \quad abc = \frac{1}{6} a^A{}_B b^{BC} c_{CA} \quad (4.10)$$

with similar expressions for b^3 and c^3 . Notice that the terms like a^3 produce just the determinant of the corresponding 3×3 matrix. Since each determinant contributes 6 terms, altogether we have 18 terms from the first three terms in (4.9). The fourth term contains 27 terms hence altogether I_3 contains precisely 45 terms. This observation will be of importance for setting up a finite geometric interpretation [19] of the structure of I_3 .

The qutrits giving rise to this nice interpretation are again real. After quantization the amplitudes a, b and c are integer, and the cubic invariant I_3 is an $E_{6(6)}(\mathbb{Z})$ invariant.

4.3. The cubic invariant, qutrits and generalized quadrangles

A *finite generalized quadrangle* of order (s, t) , usually denoted $\mathrm{GQ}(s, t)$, is an incidence structure $S = (P, B, \mathrm{I})$, where P and B are disjoint (non-empty) sets of objects, called respectively points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms [90]: (i) each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line; (ii) each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one common point; and (iii) if x is a point and L is a line not incident with x , then there exists a unique pair $(y, M) \in P \times B$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$; from these axioms it readily follows that $|P| = (s + 1)(st + 1)$ and $|B| = (t + 1)(st + 1)$.

Given two points x and y of S one writes $x \sim y$ and says that x and y are collinear if there exists a line L of S incident with both. For any $x \in P$ denote $x^\perp = \{y \in P | y \sim x\}$ and note that $x \in x^\perp$; obviously, $|x^\perp| = 1 + s + st$. Given an arbitrary subset A of P , the *perp*-(set) of A , A^\perp , is defined as $A^\perp = \bigcap \{x^\perp | x \in A\}$.

Here we shall be concerned with generalized quadrangles having lines of size *three*, $\mathrm{GQ}(2, t)$. From the above-given restrictions one readily sees that these are of three distinct kinds, namely $\mathrm{GQ}(2, 1)$, $\mathrm{GQ}(2, 2)$ and $\mathrm{GQ}(2, 4)$. $\mathrm{GQ}(2, 1)$ is a grid of 9 points on 6 lines. $\mathrm{GQ}(2, 2)$ is the smallest thick generalized quadrangle, also known as the “doily.” It is the pentagon-like object shown within Figure 4. The pairs of numbers clearly show its duad construction. This quadrangle is endowed with 15 points/lines, with each line containing 3 points and, dually, each point being on 3 lines. The last case in the hierarchy is $\mathrm{GQ}(2, 4)$, which possesses 27 points and 45 lines, with lines of size 3 and 5 lines through a point. One of its constructions goes as follows. One starts with the duad construction of $\mathrm{GQ}(2, 2)$, adds 12 more points labelled simply as $1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'$ and defines 30 additional lines as the three-sets $\{a, b', \{a, b\}\}$ of points, where $a, b \in \{1, 2, 3, 4, 5, 6\}$ and $a \neq b$. This process is diagrammatically illustrated, after Polster [91], in Figure 4.

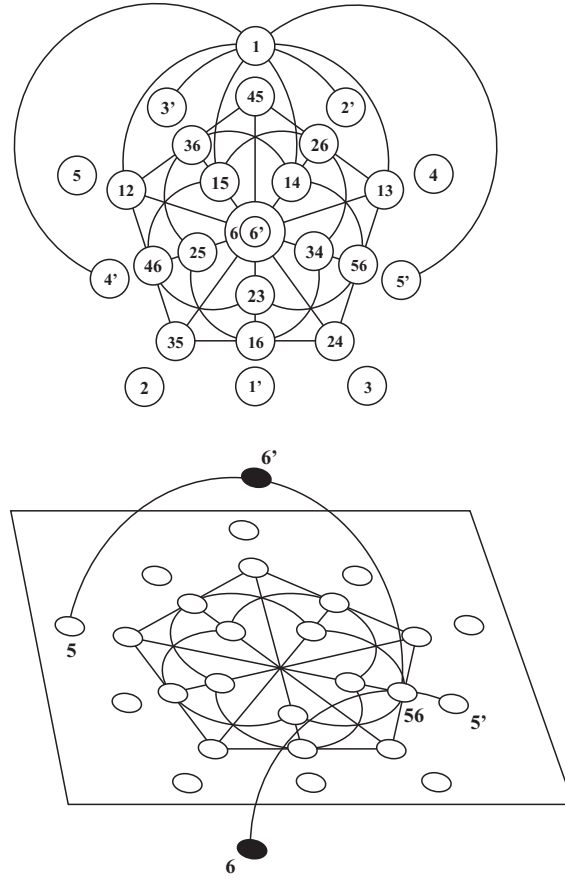


FIG. 4: A diagrammatic illustration of the structure of the generalized quadrangle $GQ(2, 4)$ after Polster [91]. In both the figures, each picture depicts all 27 points (circles). The top picture shows only 19 lines (line segments and arcs of circles) of $GQ(2, 4)$, with the two points located in the middle of the doily being regarded as lying one above and the other below the plane the doily is drawn in. 16 out of the missing 26 lines can be obtained by successive rotations of the figure through 72 degrees around the center of the pentagon. The bottom picture shows a couple of lines which go off the doily's plane; the remaining 8 lines of this kind are again got by rotating the figure through 72 degrees around the center of the pentagon.

The structure of the generalized quadrangle $GQ(2, 4)$ nicely encapsulates the structure of the cubic invariant I_3 up to signs [19]. Recall that the number of lines (45) matches the number of terms in the explicit expression of the cubic invariant of (4.9). Writing out explicitly I_3 one can deduce the labelling for $GQ(2, 4)$ described in Figure 5.

Notice that the three two-qutrit states of (4.8) partition the 27 points of $GQ(2, 4)$ to 3 disjoint grids, i. e. $GQ(2, 1)$ s. The 27 lines corresponding to the terms of $\text{Tr}(abc)$ of (4.9) are of the type like the one $a^1_2 b^{22} c_{21}$, and the $3 \times 6 = 18$ terms are coming from the three 3×3 determinants a^3, b^3, c^3 . These terms are of the form $b^{20} b^{02} b^{11}$. One can check that each of 45 lines of $GQ(2, 4)$ correspond to exactly one monomial of (4.9).

It is well-known that the automorphism group of the generalized quadrangle $GQ(2, 4)$ is the Weyl group [90] $W(E_6)$ of order 51840. This group is a subgroup of the U-duality group $E_{6(6)}(\mathbb{Z})$. For an explicit realization of this subgroup in a quantum information theoretic setting see [19]. The cubic invariant is also connected to the geometry of smooth (non-singular) cubic surfaces [82]. It was Elie Cartan who first realized [92] that the 45 monomials of our cubic form stabilized by $W(E_6)$ are in correspondence with the tritangent planes of the cubic.

4.4. Geometric hyperplanes and truncations

A *geometric hyperplane* H of a point-line geometry $\Gamma(P, B)$ is a proper subset of P such that each line of Γ meets H in one or all points [53]. The only type of hyperplanes featured in $GQ(2, 4)$ are doilies (we have 36 of them) and perp sets (their number is 27). Moreover, $GQ(2, 4)$ also contains $3 \times 40 = 120$ grids. However, these are *not* its geometric hyperplanes [93].

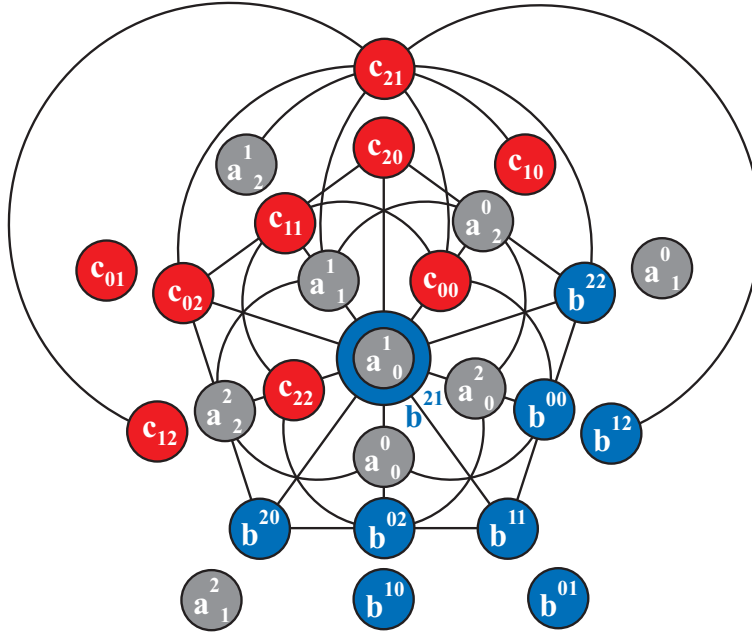


FIG. 5: A qutrit labelling of the points of $GQ(2, 4)$. Three different colours (online only) are used to illustrate a triple of grids partitioning the point set.

(This is quite different from the $GQ(2, 2)$ case, where grids are geometric hyperplanes.) Though they are not hyperplanes, they have an important property: there exists 40 triples of them, each partitioning the point set of $GQ(2, 4)$.

A decomposition of $E_{6(6)}$ directly related to a doily as a geometric hyperplane sitting inside $GQ(2, 4)$ is the following one [59, 94]

$$E_{6(6)} \supset \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(6, \mathbb{R}) \quad (4.11)$$

under which

$$\mathbf{27} \rightarrow (\mathbf{2}, \mathbf{6}) \oplus (\mathbf{1}, \mathbf{15}). \quad (4.12)$$

One can show that under this decomposition I_3 schematically factors as

$$I_3 = \mathrm{Pf}(A) + u^T A v, \quad (4.13)$$

where u and v are two six-component vectors and $\mathrm{Pf}(A)$ is the Pfaffian of an antisymmetric 6×6 matrix with 15 independent components. Clearly the decomposition above featuring the doily is nicely mapped to the duad construction of $GQ(2, 4)$, see Figure 4.

The next important type of subconfiguration of $GQ(2, 4)$ is the grid. The decomposition underlying this type of subconfiguration is the one given by (4.6). It is also obvious that the 40 triples of pairwise disjoint grids are intimately connected to the 40 different ways we can obtain a qutrit description of I_3 . Note that there are 10 grids which are geometric hyperplanes of a particular copy of the doily of $GQ(2, 4)$. This is related to the fact that the quaternionic magic case with 15 charges can be truncated to the 9 charge complex case.

The second type of hyperplanes we should consider are perp-sets. Perp-sets are obtained by selecting an arbitrary point and considering all the points collinear with it. Since we have five lines through a point, any perp set has $1 + 10 = 11$ points. A decomposition which corresponds to perp-sets is thus of the form [59]

$$E_{6(6)} \supset \mathrm{SO}(5, 5) \times \mathrm{SO}(1, 1) \quad (4.14)$$

under which

$$\mathbf{27} \rightarrow \mathbf{16}_1 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_4. \quad (4.15)$$

This is the usual decomposition of the U-duality group into the T duality and S duality. It is interesting to see that the last term (i. e. the one corresponding to the fixed/central point in a perp-set) describes the NS five-brane charge. Notice that we have five

lines going through this fixed point of a perp-set. These correspond to the T^5 used to compactify type II string theory to five dimensions. The two remaining points on each of these 5 lines correspond to $2 \times 5 = 10$ charges. They correspond to the 5 directions of KK momentum and the 5 directions of fundamental string winding. In this picture the 16 charges *not belonging to* the perp-set correspond to the 16 D-brane charges. Notice that we can get 27 similar truncations based on the 27 possible central points of the perp-set.

4.5. Three-qubit operators and the cubic invariant

In the previous subsections we managed to understand the structure of the cubic invariant giving rise to the black hole entropy of five dimensional black holes and black strings in terms of qutrits. Now we show that interestingly there is a dual way of understanding I_3 using the *real three-qubit Pauli group* [19]. This way of looking at I_3 provides a geometric framework for understanding the connection between the $d = 4$ and $d = 5$ duality groups i.e. $E_{7(7)}(\mathbb{Z})$ and $E_{6(6)}(\mathbb{Z})$.

Let us define the *real* three qubit Pauli operators by introducing the notation [18] $X \equiv \sigma_1$, $Y = i\sigma_2$, and $Z \equiv \sigma_3$; here, σ_j , $j = 1, 2, 3$ are the usual 2×2 Pauli matrices. Then we can define the real operators of the three-qubit Pauli group [4, 54, 58] by forming the tensor products of the form $ABC \equiv A \otimes B \otimes C$ that are 8×8 matrices. All possible combinations of these operators of the form $\pm ABC$ make up the real Pauli group, a set of 128 matrices endowed with the usual matrix multiplication. Notice that operators containing an even number of Y s are symmetric and those containing an odd number of Y s are antisymmetric. From the 64 possible combinations of the form ABC we have 36 symmetric matrices and 28 antisymmetric.

From the set of antisymmetric matrices let us chose the seven element subset

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7) = (IIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ) \quad (4.16)$$

satisfying the relation $\{g_a, g_b\} = -2\delta_{ab}$, $a, b = 1, 2, \dots, 7$. These operators form the generators of a seven-dimensional Clifford algebra. The remaining 21 antisymmetric operators are of the form $\frac{1}{2}[g_a, g_b]$. They generate an $\mathfrak{so}(7)$ algebra. We relate these matrices to the generators of $\text{SO}(8)$ using

$$-\Gamma^{0a} = g_{0a} \equiv g_a, \quad -\Gamma^{ab} = g_{ab} \equiv \frac{1}{2}[g_a, g_b]. \quad (4.17)$$

We can make use of these three-qubit operators for expanding the $\mathcal{N} = 8$ central charge \mathcal{Z}_{AB} as in (3.5).

Note that the decomposition

$$E_{7(7)} \supset E_{6(6)} \times \text{SO}(1, 1) \quad (4.18)$$

under which

$$\mathbf{56} \rightarrow \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{27}' \oplus \mathbf{1}' \quad (4.19)$$

describes the relation between the $D = 4$ and $D = 5$ duality groups [77, 87, 95–98]. In order to connect the qutrit and three-qubit operator pictures we assign to one of the three-qubit operators a special status

$$\Omega = IIY = g_1. \quad (4.20)$$

Now we use the $\mathcal{N} = 8$ central charge parametrized as in (3.5) and look at the structure of the cubic invariant. It can be also be written in the alternative form [78]

$$I_3 = -\frac{1}{48} \text{Tr}(\Omega \mathcal{Z} \Omega \mathcal{Z} \Omega \mathcal{Z}). \quad (4.21)$$

In order to get the correct number of components, we impose the constraints [85]

$$\text{Tr}(\Omega \mathcal{Z}) = 0, \quad \bar{\mathcal{Z}} = \Omega \mathcal{Z} \Omega^T. \quad (4.22)$$

The first of these restricts the number of antisymmetric matrices to be considered in the expansion of \mathcal{Z} from 28 to 27. The second constraint is the usual reality condition which restricts the 27 *complex* expansion coefficients to 27 *real* ones. The group theoretical meaning of these constraints is the expansion of the $\mathcal{N} = 8$ central charge in an $\text{USp}(8)$ basis, which is appropriate since $\text{USp}(8)$ is the automorphism group of the $\mathcal{N} = 8$, $D = 5$ supersymmetry algebra.

It is easy to see that the reality constraint yields for $\Omega \mathcal{Z}$ the form

$$\Omega \mathcal{Z} = \mathcal{S} + i\mathcal{A} \equiv x^{jk} g_{1jk} + i(y_{0j} g_{1j} - y_{1j} g_j). \quad (4.23)$$

Performing standard manipulations, we get

$$I_3 = -\frac{1}{48} (\text{Tr}(SSS) - 3\text{Tr}(\mathcal{S}\mathcal{A}\mathcal{A})). \quad (4.24)$$

Hence, with the notation

$$A^{jk} \equiv x^{j+1k+1}, \quad u_j \equiv y_{1j+1}, \quad v_j \equiv y_{0j+1}, \quad j, k = 1, 2, \dots, 6, \quad (4.25)$$

the terms of (4.24) give rise to the form of (4.13). Notice also that the parametrization

$$u^T = (-c_{21}, -a^2_1, -b^{01}, -a^0_1, c_{01}, b^{21}), \quad v^T = (b^{10}, -c_{10}, a^1_2, c_{12}, b^{12}, a^1_0) \quad (4.26)$$

$$A = \begin{pmatrix} 0 & c_{02} & b^{22} & -c_{00} & a^1_1 & b^{02} \\ c_{02} & 0 & a^0_0 & b^{11} & c_{22} & -a^0_2 \\ -b^{22} & -a^0_0 & 0 & a^2_0 & b^{20} & c_{11} \\ c_{00} & -b^{11} & -a^2_0 & 0 & c_{20} & a^2_2 \\ -a^1_1 & -c_{22} & -b^{20} & -c_{20} & 0 & -b^{00} \\ -b^{02} & a^0_2 & -c_{11} & -a^2_2 & b^{00} & 0 \end{pmatrix} \quad (4.27)$$

yields for I_3 its qutrit version of (4.9).

4.6. Mermin squares

At this point it is instructive to have a look again at the finite geometric structure of I_3 . The careful reader might have noticed that there is one important issue we have not clarified yet. We have established a connection between the qutrit interpretation and the structure of the generalized quadrangle $GQ(2, 4)$. However, our labelling of the points of $GQ(2, 4)$ by the real 3×3 matrices a, b and c serving as qutrit amplitudes did not manage to take care of the *signs* of the 45 terms showing up in I_3 . It is easy to see that no distribution of charges for these amplitudes is available matching the structure of I_3 and the incidence structure of $GQ(2, 4)$ at the same time.

The reason for this is very simple. According to Figure 5 the points of $GQ(2, 4)$ can be split into three grids. Moreover, according to (4.9) the relevant part of I_3 answering a particular grid is just the 3×3 determinant of the corresponding two-qutrit state. The structure of this determinant is encapsulated in the structure of the corresponding grid. We can try to arrange the 9 amplitudes in a way that the 3 plus signs for the determinant should occur along the rows and the 3 minus signs along the columns. But this is impossible since multiplying all of the nine signs “row-wise” yields a plus sign, but “column-wise” yields a minus sign.

Readers familiar with Bell-Kochen-Specker type theorems ruling out noncontextual hidden variable theories may immediately suggest that if we have failed to associate signs with the points of the grid, what about trying to use noncommutative objects instead? More precisely, we can try to associate objects that are generally noncommuting but that are pairwise commuting along the lines of the grid. This is exactly what is achieved by using Mermin squares [56, 99, 100]. Mermin squares are obtained by assigning pairwise commuting *two-qubit Pauli matrices* to the lines of the grid in such a way that the naive sign assignment does not work, but we get the identity operators with the correct signs by multiplying the operators row- and column-wise.

However, we have merely 16 real two-qubit Pauli operators up to sign, which is simply not enough to label the 27 points of our $GQ(2, 4)$. Hence we are forced to try the next item in the line: namely some subset of the real three-qubit Pauli group. Let us recall the duad labelling of $GQ(2, 4)$ as discussed in Figure 4. According to this a natural *noncommutative labelling* for the 27 points of $GQ(2, 4)$ is the following. Let us remove the special operator Ω from the 28 antisymmetric ones. Then set up the correspondence between the points of Figure 4 and the remaining operators as

$$\{1', 2', 3', 4', 5', 6'\} \leftrightarrow \{g_2, g_3, g_4, g_5, g_6, g_7\}, \quad \{1, 2, 3, 4, 5, 6\} \leftrightarrow \{g_{12}, g_{13}, g_{14}, g_{15}, g_{16}, g_{17}\} \quad (4.28)$$

$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\} \leftrightarrow \{g_{23}, g_{24}, g_{25}, g_{26}, g_{27}, g_{34}, g_{35}, g_{36}, g_{37}\} \quad (4.29)$$

$$\{34, 35, 36, 45, 46, 56\} \leftrightarrow \{g_{45}, g_{46}, g_{47}, g_{56}, g_{57}, g_{67}\} \quad (4.30)$$

i. e., by shifting all the indices of g_{IJ} not containing 0 or 1 by -1 we get the duad labels.

However, in order for this noncommutative labelling of $GQ(2, 4)$ to represent a generalization of a Mermin square: (i) the operators on each line should be pairwise commuting and (ii) at the same time their products (not depending on the order) should

5. *STU* BLACK HOLES AND ATTRACTORS.

5.1. The *STU* model

In the previous section we saw how the structure of the macroscopic Bekenstein-Hawking entropy is encoded into entanglement measures of several qubits and qutrits. Apart from the BHQC contributing to our understanding of structural issues concerning black-hole entropy in quantum information theoretic terms, the desire for an entanglement based understanding for issues of dynamics also arose. In this section we would like to discuss results connected to the dynamics of the well-known attractor mechanism [24–26] in the special case of the *STU* model [27, 28].

Theories such as the *STU* model arise in string theory, whose low energy limit is described by two derivative supergravity theories involving massless fields of spins ≤ 2 . We consider the effective action for these fields to leading order in the string coupling constant and the inverse tension. Since these string theories (and M-theory) live in ten (eleven) spacetime dimensions we have to deduce the four dimensional massless spectrum by compactification of the extra dimensions. This process is effected by decomposing the 10-dimensional fields according to harmonic forms determined by the cohomology of the extra dimensions. Thus, the geometric data of these spaces gives rise to extra fields in the low energy effective four dimensional theory. Among these fields especially important are the so-called moduli, massless scalar fields describing the fluctuation of the shape and size of the extra dimensions. The space of deformation parameters of “size and shape” is called the moduli space. The scalar fields in the four-dimensional spacetime manifold take values in this space.

The massless spectrum of string theories also contains Maxwell-like fields described by differential forms. Like the familiar Maxwell field, which is a one-form coupled to the world-line of point-like objects, these $(p+1)$ -form fields couple to the world-volumes of extended objects called p -branes. In the low energy effective four-dimensional picture such objects also give rise to Maxwell fields ($U(1)$ gauge fields) with their couplings depending on the scalar (moduli) fields.

As we have already mentioned the low energy four-dimensional actions are supergravity theories implying that accompanying the *bosonic* fields, namely the metric, the scalar fields and the Maxwell fields, are their *fermionic* counterparts. Supersymmetry transformations mix the fermionic and bosonic fields. Since our main concern here is finding the classical black hole solutions we can restrict our attention merely to the bosonic part of the Lagrangian by setting the fermionic fields to zero. Nevertheless we shall be primarily concerned with solutions that preserve some fraction of the supersymmetry. In other words they admit covariantly constant (Killing) spinors. These solutions will be called *supersymmetric* or BPS (Bogomolny, Prasad, Summerfield).

The *STU* model is a rigid $\mathcal{N} = 2$ supergravity model in $D = 4$, coupled to three vector multiplets. The $D = 4, \mathcal{N} = 2$ supergravity multiplet contains the metric (graviton), two spin-3/2 fermions (gravitini) and one spin-1 gauge potential A_μ , the so-called “graviphoton”. The vector multiplets each consist of one gauge potential, two spin-1/2 fermions (gaugini) and a single complex scalar field. In summary, the bosonic sector is precisely of the form we discussed above. Namely it contains the four-dimensional spacetime metric $g_{\mu\nu}$, three complex scalar fields $z^j, j = 1, 2, 3$, and 4 $U(1)$ vector fields A_μ^I with field strengths $\mathcal{F}_{\mu\nu}^I$ where $I = 0, 1, 2, 3$. Sometimes the three complex scalars are denoted by the letters S, T and U , hence the name of the model.

There are a number of different ways of obtaining the *STU* model from string or M-theory compactifications [27]. For example, when type IIA string theory is compactified on a six torus T^6 (or equivalently when M-theory is compactified on a T^7) one recovers $\mathcal{N} = 8$ supergravity in $D = 4$ with 28 vectors and 70 scalars. The moduli space is the coset space $E_{7(7)}/SU(8)$. This theory with an on shell U -duality symmetry $E_{7(7)}$ is already familiar from our considerations of the tripartite entanglement of seven qubits. There we saw that the *STU* model is a consistent $\mathcal{N} = 2$ truncation of this $\mathcal{N} = 8$ model. We have seven equivalent *STU* truncations corresponding to the seven points of the dual Fano plane. One may also obtain the *STU* model directly by orbifold compactification [68]. This version comes with an additional four hypermultiplets and is the one obtained by truncating the Fano plane from 7 lines to one. These hypermultiplets will not play a role in the present paper, however.

It is therefore not surprising that the study of *STU* black hole solutions is very important. For example, the single-center 1/2-BPS solutions of the *STU* model with non-zero Bekenstein-Hawking entropy may be embedded in the 1/8-BPS solutions with non-zero Bekenstein-Hawking entropy of the $\mathcal{N} = 8$ model. This implies that in order to generate the most general solution one has to act with an $SU(8)$ transformation rotating the $4+4$ charges of the *STU* model associated with the 4 $U(1)$ gauge fields to the $28+28$ charges of the $\mathcal{N} = 8$ theory. In the language of group theory this process is encapsulated in (3.27). The charges of the *STU* model correspond to a singlet of $SO(4, 4)$ and transform as a three-qubit state, i.e. a $(2, 2, 2)$ under $[SL(2, \mathbb{Z})]^{\otimes 3}$. Although in the *STU* model we have just 6 real (3 complex) scalars one can generate generic values of the scalars via applying an $E_{7(7)}$ transformation. Thus in many ways, the *STU* model serves as a basic building block.

5.2. *STU* black holes as four-qubit systems

5.2.1. Timelike dimensional reduction of the *STU* Lagrangian

The bosonic part of the action of the *STU* model is

$$\begin{aligned} \mathcal{S} = & \frac{1}{16\pi} \int d^4x \sqrt{|g|} \left\{ -\frac{R}{2} + G_{i\bar{j}} \partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}} g^{\mu\nu} \right. \\ & \left. + (\text{Im}\mathcal{N}_{IJ} \mathcal{F}^I \cdot \mathcal{F}^J + \text{Re}\mathcal{N}_{IJ} \mathcal{F}^I \cdot \star \mathcal{F}^J) \right\} \end{aligned} \quad (5.1)$$

Here $\star \mathcal{F}$ refers to the Hodge-dual of the two-form \mathcal{F} and $\mathcal{F} \cdot \mathcal{F} = \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$. The manifold of the scalar fields for the *STU* model is $[\text{SL}(2, \mathbb{R}) / \text{U}(1)]^{\otimes 3}$. In the following we will denote the three complex scalar fields as

$$z^j \equiv x^j - iy^j, \quad j = 1, 2, 3, \quad y^j > 0. \quad (5.2)$$

With these definitions the metric on the scalar manifold (moduli space) is

$$G_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{(2y^i)^2}. \quad (5.3)$$

The metric above can be derived from the Kähler potential

$$K = -\log(8y_1 y_2 y_3) \quad (5.4)$$

as $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. For the *STU* model the scalar dependent vector couplings $\text{Re}\mathcal{N}_{IJ}$ and $\text{Im}\mathcal{N}_{IJ}$ take the following form

$$\nu_{IJ} \equiv \text{Re}\mathcal{N}_{IJ} = \begin{pmatrix} 2x_1 x_2 x_3 & -x_2 x_3 & -x_1 x_3 & -x_1 x_2 \\ -x_2 x_3 & 0 & x_3 & x_2 \\ -x_1 x_3 & x_3 & 0 & x_1 \\ -x_1 x_2 & x_2 & x_1 & 0 \end{pmatrix} \quad (5.5)$$

$$\mu_{IJ} \equiv \text{Im}\mathcal{N}_{IJ} = -y_1 y_2 y_3 \begin{pmatrix} 1 + \left(\frac{x_1}{y_1}\right)^2 + \left(\frac{x_2}{y_2}\right)^2 + \left(\frac{x_3}{y_3}\right)^2 & -\frac{x_1}{y_1^2} & -\frac{x_2}{y_2^2} & -\frac{x_3}{y_3^2} \\ -\frac{x_1}{y_1^2} & \frac{1}{y_1^2} & 0 & 0 \\ -\frac{x_2}{y_2^2} & 0 & \frac{1}{y_2^2} & 0 \\ -\frac{x_3}{y_3^2} & 0 & 0 & \frac{1}{y_3^2} \end{pmatrix} \quad (5.6)$$

Our aim is to describe stationary solutions of the Euler-Lagrange equations arising from the *STU* action (5.1) in an entanglement based language.

It is well-known that the most general ansatz for stationary solutions in four dimensions is [102]

$$ds^2 = -e^{2U} (dt + \omega)^2 + e^{-2U} h_{ab} dx^a dx^b \quad (5.7)$$

$$\mathcal{F}^I = d\mathcal{A}^I = d(\xi^I (dt + \omega) + A^I) \quad (5.8)$$

where $a, b = 1, 2, 3$ correspond to the spacial directions. The quantities $U, \xi^I, A_a^I, \omega_a$ and h_{ab} are regarded as 3D fields, i.e. the ansatz above corresponds to dimensional reduction to $D = 3$ along the timelike direction. In achieving this we have chosen the gauge such that the Lie-derivative of \mathcal{A}^I with respect to the timelike Killing vector vanishes, and have chosen coordinates such that the isometry corresponding to this Killing vector is just a (time) translation. In this case the quantities in (5.7-5.8) depend only on $x^a, a = 1, 2, 3$.

After performing the dimensional reduction to $D = 3$ our starting Lagrangian of (5.1) takes the following form [102, 103]

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \quad (5.9)$$

where

$$\mathcal{L}_1 = -\frac{1}{2} \sqrt{h} R[h] + dU \wedge \star dU + \frac{1}{4} e^{-4U} (d\sigma + \tilde{\xi}_I d\xi^I - \xi^I d\tilde{\xi}_I) \wedge \star (d\sigma + \tilde{\xi}_J d\xi^J - \xi^J d\tilde{\xi}_J) \quad (5.10)$$

$$\mathcal{L}_2 = G_{i\bar{j}} dz^i \wedge \star d\bar{z}^{\bar{j}} \quad (5.11)$$

$$\mathcal{L}_3 = \frac{1}{2} e^{-2U} \mu_{IJ} d\xi^I \wedge \star d\xi^J + \frac{1}{2} e^{-2U} \mu^{IJ} (d\tilde{\xi}_I - \nu_{IK} d\xi^K) \wedge \star (d\tilde{\xi}_J - \nu_{JL} d\xi^L). \quad (5.12)$$

Here the new (axionic) scalars σ and $\tilde{\xi}_I$ come from dualizing ω and A^I by [102]

$$d\tilde{\xi}_I \equiv \nu_{IJ} d\xi^J - e^{2U} \mu_{IJ} \star (dA^J + \xi^J d\omega) \quad (5.13)$$

$$d\sigma \equiv e^{4U} \star d\omega + \xi^I d\tilde{\xi}_I - \tilde{\xi}_I d\xi^I. \quad (5.14)$$

Note also that here the exterior derivative is understood on the spatial slice with local coordinates x^a , $a = 1, 2, 3$.

The dimensionally reduced Lagrangian \mathcal{L} can be written in the nice form of 3-dimensional gravity coupled to a nonlinear sigma model defined on the spatial slice with target manifold [104] $\mathcal{M}_3 = \text{SO}(4, 4)/[\text{SL}(2, \mathbb{R})]^{\otimes 4}$ with the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \sqrt{h} R[h] + g_{mn} \partial_a \Phi^m \partial^a \Phi^n \quad (5.15)$$

where Φ^m , $m = 1, 2, \dots, 16$ refers to the scalar fields: $U, \sigma, \xi^I, \tilde{\xi}_I, z^j, \bar{z}^{\bar{j}}$ with $I = 0, 1, 2, 3$ and $j = 1, 2, 3$. Here the line element on \mathcal{M}_3 defines g_{mn} as $ds_{\mathcal{M}_3}^2 = g_{mn} d\Phi^m d\Phi^n$ with the explicit form

$$\begin{aligned} \frac{1}{4} ds_{\mathcal{M}_3}^2 &= G_{i\bar{j}}(z, \bar{z}) dz^i d\bar{z}^{\bar{j}} + dU^2 + \frac{1}{4} e^{-4U} (d\sigma + \tilde{\xi}_I d\xi^I - \xi^I d\tilde{\xi}_I)^2 \\ &+ \frac{1}{2} e^{-2U} \left[\mu_{IJ} d\xi^I d\xi^J + \mu^{IJ} (d\tilde{\xi}_I - \nu_{IK} d\xi^K) (d\tilde{\xi}_J - \nu_{JL} d\xi^L) \right]. \end{aligned} \quad (5.16)$$

5.2.2. The line element as a four-qubit measure

We have seen that in the 3D picture the moduli space is the coset $\mathcal{M}_3 = \text{SO}(4, 4)/\text{SL}(2, \mathbb{R})^{\otimes 4}$. Due to the presence of $\text{SL}(2, \mathbb{R})^{\otimes 4}$ which is a subgroup of the real SLOCC group $\text{GL}(2, \mathbb{R})^{\otimes 4}$ one should be tempted to try a four-qubit reformulation incorporating all the quantities of the *STU* model.

In order to do this note that our coset can be locally parametrized by 16 independent quantities. These are the 6 quantities (x_j, y_j) , $j = 1, 2, 3$ coming from the scalar fields of (5.2), the 8 potentials $\xi^I, \tilde{\xi}_I$, the NUT potential [105] σ defined by (5.14) and the warp factor U showing up in the metric ansatz of (5.7). We introduce new quantities

$$\zeta^I \equiv \sqrt{2} \xi^I, \quad \tilde{\zeta}_I \equiv \sqrt{2} \tilde{\xi}_I, \quad x_0 \equiv \sigma, \quad y_0 \equiv e^\phi = e^{2U}. \quad (5.17)$$

Then in the Iwasawa parametrization [32] we can describe our coset by the matrix \mathcal{V}

$$\mathcal{V} = \begin{pmatrix} M_3 \otimes M_2 & 0 \\ 0 & M_1 \otimes M_0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & -\zeta g \\ \zeta^T g & \mathbf{1} + \frac{1}{2} \Delta \end{pmatrix}. \quad (5.18)$$

Here

$$M_\alpha \equiv \frac{1}{\sqrt{y_\alpha}} \begin{pmatrix} 1 & -x_\alpha \\ 0 & y_\alpha \end{pmatrix} \quad \alpha = 0, 1, 2, 3 \quad (5.19)$$

$$\Delta = \begin{pmatrix} \zeta^{(0)} \cdot \zeta^{(0)} & \zeta^{(0)} \cdot \zeta^{(1)} \\ \zeta^{(0)} \cdot \zeta^{(1)} & \zeta^{(1)} \cdot \zeta^{(1)} \end{pmatrix} \varepsilon \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (5.20)$$

Here the $\text{SL}(2) \times \text{SL}(2)$ invariant \cdot product of two four component vectors is defined with respect to the 4×4 matrix $g = \varepsilon \otimes \varepsilon$, with ε defined as in (3.8). The 4-component vectors $\zeta^{(0)}$ and $\zeta^{(1)}$ are just the first and third columns of the matrix ζ defined as

$$\zeta \equiv \begin{pmatrix} \zeta_{0000} & \zeta_{0001} & \zeta_{0010} & \zeta_{0011} \\ \zeta_{0100} & \zeta_{0101} & \zeta_{0110} & \zeta_{0111} \\ \zeta_{1000} & \zeta_{1001} & \zeta_{1010} & \zeta_{1011} \\ \zeta_{1100} & \zeta_{1101} & \zeta_{1110} & \zeta_{1111} \end{pmatrix} = \begin{pmatrix} -\tilde{\zeta}_0 & 0 & \tilde{\zeta}_1 & 0 \\ \tilde{\zeta}_2 & 0 & \zeta^3 & 0 \\ \tilde{\zeta}_3 & 0 & \zeta^2 & 0 \\ \zeta^1 & 0 & \zeta^0 & 0 \end{pmatrix} \quad (5.21)$$

and $\mathbf{1}$ is the 4×4 identity matrix.

Using the coset representative \mathcal{V} the line element on \mathcal{M}_3 is given by the formula [32, 104]

$$ds^2 = \text{Tr}(P)^2 \quad (5.22)$$

where

$$P \equiv \frac{1}{2}(d\mathcal{V}\mathcal{V}^{-1} + \eta(d\mathcal{V}\mathcal{V}^{-1})^T \eta) \quad (5.23)$$

and the involution compatible with our conventions is

$$\eta = \begin{pmatrix} I \otimes I & 0 \\ 0 & -I \otimes I \end{pmatrix}. \quad (5.24)$$

Let us introduce a four-qubit state which is a differential form on the symplectic torus determined by the Wilson lines

$$|\Psi\rangle = (M_3 \otimes M_2 \otimes M_1 \otimes M_0)|d\zeta\rangle. \quad (5.25)$$

Using this we obtain for the line element on \mathcal{M}_3 the following form

$$ds_{\mathcal{M}_3}^2 = \sum_{j=1}^3 \frac{dx_j^2 + dy_j^2}{y_j^2} + \frac{(dx_0 - w)^2 + dy_0^2}{y_0^2} - ||\Psi||^2 \quad (5.26)$$

where $||\Psi||^2 \equiv \langle \Psi | \Psi \rangle$, and

$$w = \frac{1}{2}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I). \quad (5.27)$$

Looking at the expression at (5.25) we see that $|\Psi\rangle$ is on the *real* SLOCC orbit of $|d\zeta\rangle$ which determines the orbit type. It is useful to embed this real state in a *complex* state, using the Hadamard gate appearing in (3.15) and the phase gate

$$P = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.28)$$

by defining

$$|\hat{\Psi}\rangle = (H \otimes H \otimes H \otimes H)(P \otimes P \otimes P \otimes P)(M_3 \otimes M_2 \otimes M_1 \otimes M_0)|d\zeta\rangle. \quad (5.29)$$

Note, this new 4-qubit state is now on the SLOCC, i.e. $[\text{GL}(2, \mathbb{C})]^{\otimes 4}$, orbit. It can be shown [32] that the amplitudes of this state depend only on the following four quantities and their conjugates

$$\mathcal{E}_0 = \sqrt{2}e^{\frac{K}{2}-U} X^I (\mathcal{N}_{IJ} d\zeta^J - d\tilde{\zeta}_I), \quad \mathcal{E}_j = 2i\sqrt{2}y_j e^{-U} f_j^I (\bar{\mathcal{N}}_{IJ} d\zeta^J - d\tilde{\zeta}_I) \quad (5.30)$$

well known from special Kähler geometry [24, 25, 106]. Here

$$f_1^I = e^{\frac{K}{2}} D_1 X^I = e^{\frac{K}{2}} (\partial_1 + (\partial_1 K)) X^I = e^{\frac{K}{2}} \frac{1}{\bar{z}_1 - z_1} \begin{pmatrix} 1 \\ \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix}, \quad (5.31)$$

where $X^I = (1, z_1, z_2, z_3)^T$, $K = -\log(y_1 y_2 y_3)$ and \mathcal{N}_{IJ} is defined by (5.5-5.6). The line element is then given by,

$$ds_{\mathcal{M}_3}^2 = \sum_{\alpha=0}^3 (\bar{e}_\alpha e_\alpha - \bar{\mathcal{E}}_\alpha \mathcal{E}_\alpha) = \sum_{\alpha=0}^3 \frac{d\bar{z}_\alpha z_\alpha}{y_\alpha^2} - ||\hat{\Psi}||^2. \quad (5.32)$$

Here $e_\alpha = \frac{-i}{y_\alpha} dz_\alpha$ are the right invariant one forms with $dz_0 = (dx_0 - w) - idy_0$ and $dz_j = dx_j - idy_j$.

Notice that according to (5.21) the four-qubit state $|d\zeta\rangle$ which determines the orbit type of $|\hat{\Psi}\rangle$ is very special. In particular, though written in a four-qubit form, it contains merely 8 nonzero amplitudes reminiscent of a three-qubit state. This special structure is due to the special status of the fourth $\text{SL}(2, \mathbb{R})$, the so called Ehlers group [107], associated with the fourth qubit. Moreover, the only quantities which play any role in $|\hat{\Psi}\rangle$ are given by (5.30). In order to incorporate the information contained

in the right-invariant forms e_α we introduce yet another four-qubit state which already contains all 16 *real* quantities associated to our coset. Neglecting the details [32] this state is given by

$$|\Lambda\rangle = \sum_{a_3, a_2, a_1, a_0=0,1} \Lambda_{a_3 a_2 a_1 a_0} |a_3 a_2 a_1 a_0\rangle \quad (5.33)$$

with amplitudes

$$\Lambda = \begin{pmatrix} \Lambda_{0000} & \Lambda_{0001} & \Lambda_{0010} & \Lambda_{0011} \\ \Lambda_{0100} & \Lambda_{0101} & \Lambda_{0110} & \Lambda_{0111} \\ \Lambda_{1000} & \Lambda_{1001} & \Lambda_{1010} & \Lambda_{1011} \\ \Lambda_{1100} & \Lambda_{1101} & \Lambda_{1110} & \Lambda_{1111} \end{pmatrix} \equiv \begin{pmatrix} -\mathcal{E}_0 & -e_0 & -e_1 & -\bar{\mathcal{E}}_1 \\ e_2 & \bar{\mathcal{E}}_2 & \mathcal{E}_3 & \bar{e}_3 \\ e_3 & \bar{\mathcal{E}}_3 & \mathcal{E}_2 & \bar{e}_2 \\ -\mathcal{E}_1 & -\bar{e}_1 & -\bar{e}_0 & -\bar{\mathcal{E}}_0 \end{pmatrix}. \quad (5.34)$$

This state is of central importance for the considerations of the following sections. It is a *complex* four-qubit state satisfying the reality condition

$$\overline{|\Lambda\rangle} = (\sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1) |\Lambda\rangle, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (5.35)$$

where σ_1 is the bit flip gate of quantum information theory. It is straightforward to check that the subgroup of $[\text{SL}(2, \mathbb{C})]^{\otimes 4}$ preserving the reality condition is $[\text{SU}(1, 1)]^{\otimes 4}$. Hence the admissible transformations are of the form

$$|\Lambda\rangle \mapsto (S \otimes S_2 \otimes S_1 \otimes S_0) |\Lambda\rangle, \quad S_3, S_2, S_1, S_0 \in \text{SU}(1, 1). \quad (5.36)$$

The number of algebraically independent $[\text{SL}(2, \mathbb{C})]^{\otimes 4}$ invariants is four [45]. We have a quadratic, two quartic, and one sextic invariant. The structure and geometry of these invariants have been investigated in [46]. Now we observe that the quadratic four-qubit invariant [45] for our state $|\Lambda\rangle$ is precisely the line element $ds_{\mathcal{M}_3}^2$

$$ds_{\mathcal{M}_3}^2 = -\frac{1}{2} \varepsilon^{a_3 a'_3} \varepsilon^{a_2 a'_2} \varepsilon^{a_1 a'_1} \varepsilon^{a_0 a'_0} \Lambda_{a_3 a_2 a_1 a_0} \Lambda_{a'_3 a'_2 a'_1 a'_0} = \sum_{\alpha=0}^3 (\bar{e}_\alpha e_\alpha - \bar{\mathcal{E}}_\alpha \mathcal{E}_\alpha). \quad (5.37)$$

This quadratic invariant is also a permutation invariant. However, the special role we have attached to the first qubit (associated with the Ehlers group) obviously breaks this permutation invariance.

5.2.3. Conserved quantities. A three qubit reformulation.

Looking at the Lagrangian (5.15) we see that the second term describes geodesic motion on the target space $\mathcal{M}_3 = \text{SO}(4, 4)/[\text{SL}(2, \mathbb{R})]^{\otimes 4}$ with line element given by the quadratic four-qubit invariant (5.33). For pseudo-Riemann symmetric target spaces, such as $\mathcal{M}_3 = \text{SO}(4, 4)/[\text{SL}(2, \mathbb{R})]^{\otimes 4}$, stationary spherically symmetric black hole solutions can be obtained as geodesic curves on this target space. See, for example [108] and the references therein. Such geodesic curves are classified in terms of the Noether charges of the solutions. Combining these results we can relate different black hole solutions to the different SLOCC entanglement classes of four-qubits [22, 23, 32]. In order to set the stage for reviewing these results let's look at the conserved quantities related to the Noether charge \mathcal{Q} .

The 3-dimensionial U-duality group $\text{SO}(4, 4)$ of the *STU* model acts isometrically on our coset \mathcal{M}_3 by right multiplication and yields a conserved Noether charge [104, 108, 109]

$$\mathcal{Q} = \mathcal{V}^{-1} P \mathcal{V} \quad (5.38)$$

where P and \mathcal{V} are defined by (5.18) and (5.23). An analysis [32, 104] of the relevant parts of \mathcal{Q} shows that we have the following conserved quantities.

First of all we have the NUT charge

$$k = p_\sigma = \frac{dx_0 - w}{2y_0^2}. \quad (5.39)$$

Here the notation p_σ refers to the fact that when using the relevant part of the Lagrangian this quantity is canonically conjugated to $x_0 \equiv \sigma$. We also have 8 conserved quantities arranged within a conserved four-qubit state $|\Gamma\rangle$ defined as

$$|\Gamma\rangle = \frac{1}{2} e^{-2U} (\mathcal{N} \otimes I) |d\zeta\rangle - p_\sigma (\epsilon \otimes I) |\zeta\rangle \quad (5.40)$$

with

$$\mathcal{N} \equiv N_3 \otimes N_2 \otimes N_1, \quad \epsilon \equiv \varepsilon \otimes \varepsilon \otimes \varepsilon, \quad N_\alpha = M_\alpha^T M_\alpha. \quad (5.41)$$

Here we also displayed the special role of the qubit corresponding to the Ehlers group, facilitating an effective three-qubit picture. The 8 conserved components as amplitudes of a four-qubit state are arranged as

$$\Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} p^0 & 0 & -p^1 & 0 \\ -p^2 & 0 & q_3 & 0 \\ -p^3 & 0 & q_2 & 0 \\ q_1 & 0 & q_0 & 0 \end{pmatrix} \quad (5.42)$$

and are related to the usual charges of the STU model. Comparing with (5.21) we see that only the Γ_{lkj0} amplitudes are nonzero.

The momenta canonically conjugate to ζ^I and $\tilde{\zeta}_I$ [104] suggest that it is rewarding to introduce the new conserved quantity

$$|\hat{\Gamma}\rangle \equiv |\Gamma\rangle + p_\sigma(\epsilon \otimes I)|\zeta\rangle \quad (5.43)$$

One can then show that the Hamiltonian governing the dynamics of our 16 fields depending on the conserved charges is

$$H = \sum_{\alpha=0}^3 y_\alpha^2 (p_{x_\alpha}^2 + p_{y_\alpha}^2) - e^{2U} \langle \hat{\Gamma} | \mathcal{N}^{-1} \otimes I | \hat{\Gamma} \rangle \quad (5.44)$$

where $\mathcal{N} = N_1 \otimes N_2 \otimes N_3$.

For vanishing NUT charge $k = p_\sigma = 0$ the second term is

$$e^{2U} V_{BH} = e^{2U} \langle \Gamma | \mathcal{N}^{-1} \otimes I | \Gamma \rangle = e^{2U} \frac{1}{2} \begin{pmatrix} p^I & q_I \end{pmatrix} \begin{pmatrix} (\mu + \nu \mu^{-1} \nu)_{IJ} & -(\nu \mu^{-1})_I^J \\ -(\mu^{-1} \nu)_J^I & (\mu^{-1})^{IJ} \end{pmatrix} \begin{pmatrix} p^J \\ q_J \end{pmatrix} \quad (5.45)$$

which gives the usual expression for the Black Hole Potential V_{BH} .

Now using (5.40) and (5.43) one can express $|d\zeta\rangle$ hence an explicit formula for $|\hat{\Psi}\rangle$ the discrete Fourier transformed state (5.29) can be derived. Using $H P M^{T^{-1}} = V S \sigma_3$ where

$$V \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, \quad S_j \equiv \frac{1}{\sqrt{y_j}} \begin{pmatrix} y_j & 0 \\ -x_j & 1 \end{pmatrix}, \quad j = 1, 2, 3 \quad (5.46)$$

the 3-qubit part of $|\hat{\Psi}\rangle$ can be written as $|\hat{\Psi}\rangle_3 \equiv i\sqrt{2}|\hat{\chi}\rangle$ where

$$|\hat{\chi}\rangle = e^U (V \otimes V \otimes V) (S_3 \otimes S_2 \otimes S_1) |\hat{\gamma}\rangle. \quad (5.47)$$

Here by virtue of (5.43)

$$|\hat{\gamma}\rangle = (\sigma_3 \otimes \sigma_3 \otimes \sigma_3) |\Gamma\rangle + p_\sigma (\sigma_1 \otimes \sigma_1 \otimes \sigma_1) |\zeta\rangle. \quad (5.48)$$

Equations (5.46)-(5.48) constitute the final result of our investigations. These expressions show that after performing the timelike dimensional reduction of our starting Lagrangian, stationary black hole solutions can be characterized by a *complex* three-qubit state $|\hat{\chi}\rangle$ depending on the charges (electric, magnetic and NUT), the warp factor, the moduli and the potentials ζ^I and $\tilde{\zeta}_I$. For nonvanishing NUT charge the SLOCC class of this state depends on the class of $|\hat{\gamma}\rangle$ of (5.48). If we assume also spherical symmetry this class is a function of the radial coordinate. However, for vanishing NUT charge the SLOCC class is entirely determined by the constant electric and magnetic charges. Moreover, in this special case a calculation shows that

$$||\hat{\chi}||^2 = e^{2U} V_{BH} \quad (5.49)$$

i.e. for vanishing NUT charge the Black Hole Potential is given by the norm of the corresponding three-qubit state $|\psi\rangle$ obtained from $|\chi\rangle$ after removing the warp factor and putting $p_\sigma = 0$ in (5.48). Notice that though our state $|\hat{\chi}\rangle$ is complex it satisfies the *reality condition*

$$\hat{\chi}_{111} = -\hat{\chi}_{000}, \quad \hat{\chi}_{001} = -\hat{\chi}_{110}, \quad \hat{\chi}_{010} = -\hat{\chi}_{101}, \quad \hat{\chi}_{100} = -\hat{\chi}_{011}. \quad (5.50)$$

5.3. Static spherically symmetric extremal solutions

In the next sections we would like to present an entanglement based understanding of weakly extremal solutions of the STU model. These are black hole solutions for which the spacial slices provided by the metric h_{ab} of (5.7) are flat [103, 104]. Single centered black holes with spherical symmetry are of this type. In this case the dynamics of the moduli are decoupled from the $3D$ gravity and the metric ansatz can be chosen to be of the form

$$ds^2 = -e^{2U}(dt + \omega) + e^{-2U}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi)) \quad (5.51)$$

with the warp factor depending merely on r . It can be shown that now the equations of motion are equivalent to light-like geodesic motion on \mathcal{M}_3 with the affine parameter $\tau = \frac{1}{r}$. We have seen that due to the fact that \mathcal{M}_3 is a symmetric space there is a number of conserved Noether charges associated with this geodesic motion. The most important ones are the electric and magnetic charges p^I and q_I and the NUT charge k [104, 108, 109]. Static solutions are characterized by the vanishing of the NUT charge i.e. $k = 0$. In this case the dynamics is described by the Lagrangian (or equivalently by the Hamiltonian of (5.44)) of a fiducial particle in the black-hole potential V_{BH} of (5.45)

$$\mathcal{L}(U(\tau), z^a(\tau), \bar{z}^{\bar{a}}(\tau)) = \left(\frac{dU}{d\tau}\right)^2 + G_{a\bar{a}} \frac{dz^a}{d\tau} \frac{d\bar{z}^{\bar{a}}}{d\tau} + e^{2U} V_{BH}(z, \bar{z}, p, q) \quad (5.52)$$

with the constraint

$$\left(\frac{dU}{d\tau}\right)^2 + G_{a\bar{a}} \frac{dz^a}{d\tau} \frac{d\bar{z}^{\bar{a}}}{d\tau} - e^{2U} V_{BH}(z, \bar{z}, p, q) = 0. \quad (5.53)$$

Notice that the latter is just about the vanishing of the Hamiltonian of (5.44). Equivalently, the line element (5.32) is vanishing. This is just another way of saying that our black hole solutions give rise to a light-like geodesic motion on \mathcal{M}_3 . According to (5.37) our constraint is also equivalent to the vanishing of an entanglement measure for the four-qubit state of (5.33).

Later we will need an alternative expression for V_{BH} that can be given in terms of the central charge of $\mathcal{N} = 2$ supergravity [110–112]

$$V_{BH} = Z\bar{Z} + G^{i\bar{j}}(D_i Z)(\bar{D}_{\bar{j}} \bar{Z}) \quad (5.54)$$

where for the STU model [70]

$$Z = e^{K/2} W = e^{K/2} (q_0 + z_1 q_1 + z_2 q_2 + z_3 q_3 + z_1 z_2 z_3 p^0 - z_2 z_3 p^1 - z_1 z_3 p^2 - z_1 z_2 p^3), \quad (5.55)$$

D_a is the Kähler covariant derivative

$$D_i Z = (\partial_i + \frac{1}{2} \partial_i K) Z \quad (5.56)$$

and W is referred to as the *superpotential*.

Extremization of the effective Lagrangian (5.52) with respect to the warp factor and the scalar fields yields the Euler-Lagrange equations

$$\ddot{U} = e^{2U} V_{BH}, \quad \ddot{z}^i + \Gamma_{jk}^i \dot{z}^j \dot{z}^k = e^{2U} \partial^i V_{BH}. \quad (5.57)$$

In these equations the dots denote derivatives with respect to $\tau = \frac{1}{r}$. These radial evolution equations taken together with the constraint (5.53) determine the structure of static, spherically symmetric, extremal black hole solutions in the STU model. For the more general stationary case with nonvanishing NUT charge ($k \neq 0$) the motion along ζ^I , $\tilde{\zeta}_I$ and σ does not decouple from U and z^j . In this case we obtain a generalization of (5.57). We will not consider solutions of such kind hence we will not give the corresponding equations here.

We conclude that the radial evolution associated to stationary spherical symmetric black hole solutions of the $D = 4$ STU model can be described as geodesic motion in the moduli space \mathcal{M}_3 of the time-like reduced $D = 3$ theory. The four-qubit picture hinges on the enlargement of the $D = 4$ symmetry $[\text{SL}(2, \mathbb{R})]^{\otimes 3}$ to the $D = 3$ symmetry $\text{SO}(4, 4)$ containing $[\text{SL}(2, \mathbb{R})]^{\otimes 4}$ as a subgroup. We are now in a position to see how the entanglement encoded in our state $|\Lambda\rangle$ of (5.33) helps to classify static spherically symmetric extremal single-centre black hole solutions in the STU model.

5.4. Black hole solutions as entangled systems.

5.4.1. BPS solutions

Let us consider the four-qubit state $|\Lambda\rangle$ of (5.33). In this subsection we will be interested in the sufficient and necessary condition for the separability of its *first qubit*, labelled by a_0 in (5.33). From our previous considerations it is clear that this qubit has a special status. In quantum information theoretic terms separability of this qubit is *equivalent* to the condition that the (unnormalized) 2×2 reduced density matrix $\varrho_1 \equiv \text{Tr}_1 |\Lambda\rangle\langle\Lambda|$ represents a pure state. This density matrix is of the form

$$\varrho_1 = \begin{pmatrix} \langle\Lambda_0|\Lambda_0\rangle & \langle\Lambda_0|\Lambda_1\rangle \\ \langle\Lambda_1|\Lambda_0\rangle & \langle\Lambda_1|\Lambda_1\rangle \end{pmatrix}, \quad \langle\Lambda_{a_0}|\Lambda_{a'_0}\rangle \equiv \sum_{a_3, a_2, a_1=0,1} \bar{\Lambda}_{a_3 a_2 a_1 a_0} \Lambda_{a_3 a_2 a_1 a'_0}. \quad (5.58)$$

This is a pure state if and only if $\text{Det} \varrho_1 = 0$. Equivalently this condition is satisfied iff $\Lambda_{a_3 a_2 a_1 0} = \lambda \Lambda_{a_3 a_2 a_1 1}$. By virtue of the reality condition of (5.35) we also have the constraint $|\lambda| = 1$. Using the definitions in (5.34) this means that

$$\mathcal{E}_0 = \lambda e_0, \quad \mathcal{E}_j = \lambda \bar{e}_j, \quad |\lambda| = 1. \quad (5.59)$$

Clearly now the constraint of (5.53) is satisfied, equivalently the quadratic four-qubit invariant is vanishing. A calculation also shows that actually all of the four-qubit invariants are vanishing [32]. Such states are called nilpotent. It can be shown that such states gives rise to a Noether charge Q of (5.38) which is a nilpotent matrix.

In order to link these considerations to the usual BPS black hole solutions we choose λ as

$$\lambda = -i \sqrt{\frac{Z}{\bar{Z}}}. \quad (5.60)$$

In the language of supergravity the above condition on separability correspondes to the existence of Killing spinors characterizing supersymmetric solutions [104]. These considerations give rise to the well-known attractor flow equations [24–26, 104]

$$\dot{U} = -e^U |Z|, \quad \dot{z}^j = -2e^U G^{j\bar{k}} \partial_{\bar{k}} |Z|. \quad (5.61)$$

These first order equations imply that the corresponding second order equations of (5.57) also hold.

Note, for weakly extremal solutions to be also extremal we also have to ensure that the solutions are smooth. To ensure this one must fine tune the boundary conditions at spatial infinity so that the fiducial particle reaches the top of the potential hill defined by V_{BH} in infinite proper time and with zero velocity. In our case an analysis of the first order equations (5.61) shows that this indeed can be achieved [113].

From this analysis we have learnt that the condition of separability for the first qubit for the four-qubit state $|\Lambda\rangle$ taken together with the special choice of (5.60) yields the first order attractor flow equations. Moreover, in this case $|\Lambda\rangle$ is a *nilpotent* state. This property of $|\Lambda\rangle$ is related to the well-known nilpotency of the Noether charge Q [104, 108, 109]. An analysis of the explicit form of these solutions will be given in Section 5.6.2.

5.4.2. Non-BPS solutions

As our first example of non-supersymmetric solutions let us discuss the separability properties of $|\Lambda\rangle$ associated with the remaining three qubits not playing a distinguished role. Here we chose to consider separability of the fourth qubit. An argument similar to the one as given in the previous subsection shows that the sufficient and necessary condition of separability for this qubit is that the first row of (5.34) is proportional to the third and the second is proportional to the fourth. Due to the reality condition we again have $|\lambda| = 1$ and we get

$$\mathcal{E}_0 = -\lambda e_3, \quad \bar{\mathcal{E}}_1 = -\lambda \bar{e}_2, \quad \bar{\mathcal{E}}_2 = -\lambda \bar{e}_1, \quad \mathcal{E}_3 = -\lambda \bar{e}_0. \quad (5.62)$$

Using the definitions of (5.30) these conditions take the explicit form

$$\frac{\dot{z}_0}{y_0} = \bar{\lambda} e^U Z_3, \quad \frac{\dot{z}_1}{y_1} = \lambda e^U Z_2, \quad \frac{\dot{z}_2}{y_2} = \lambda e^U Z_1, \quad \frac{\dot{z}_3}{y_3} = -\bar{\lambda} e^U Z, \quad (5.63)$$

where $Z_j \equiv D_j Z$ as given by (5.56). Now for static solutions we again have no twist potential, $x_0 = 0$, hence by choosing

$$\lambda = -i \sqrt{\frac{Z_3}{Z}} \quad (5.64)$$

we get

$$\dot{U} = -e^U |Z_3|, \quad \dot{z}^j = -2e^U G^{j\bar{k}} \partial_{\bar{k}} |Z_3|. \quad (5.65)$$

These expressions show that demanding separability for the fourth qubit taken together with the choice of (5.64) yields the first order equations characterizing attractors with vanishing central charge [114].

Clearly similar considerations apply for issues of separability for the second and third qubits. The result will be similar sets of equations with $|Z_3|$ replaced by $|Z_1|$ and $|Z_2|$. This amounts to taking different forms for the so-called fake superpotential [104]. Calculations again show that the four algebraically independent four-qubit invariants are zero, hence our considerations on the nilpotency of $|\Lambda\rangle$ familiar from the previous subsection still apply.

Let us now discuss a non-BPS solution with non-vanishing central charge. Obviously the vanishing of the quadratic four-qubit invariant i.e. the (5.53) constraint can be satisfied in a number of different ways. Explicitly,

$$\sum_{\alpha=0}^3 \bar{\mathcal{E}}_\alpha \mathcal{E}_\alpha = \sum_{\alpha=0}^3 \bar{e}_\alpha e_\alpha. \quad (5.66)$$

For static solutions we have already remarked that $\bar{e}_0 = e_0$, hence for BPS solutions (5.59)-(5.60) can be written in the form $\mathcal{E}_\alpha = \lambda \bar{e}_\alpha$, i.e. \mathcal{E}_α is related to \bar{e}_α via a special element of $U(4)$ containing only phase factors λ in its diagonal. In the case of non-BPS solutions with vanishing central charge these elements of $U(4)$ are just permutation matrices combined with similar phase factors and their conjugates. This structure is related to the *separability* of one of the qubits in the state $|\Lambda\rangle$.

In order to obtain states $|\Lambda\rangle$ which are *entangled* and at the same time give rise to static spherically symmetric non-BPS black hole solutions with non-vanishing central charge let us consider the following choice

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (5.67)$$

Due to the unitarity of the relevant matrix (5.66) is satisfied moreover, one can show that none of the qubits can be separated from the rest. However, we still have to satisfy the equations of motion (5.57). Let us illustrate that the choice of (5.67) indeed gives rise to a solution of the latter equations. This solution is the non-BPS seed solution [115]. First recall the definition of the three-qubit state $|\chi\rangle$ of (5.47) and denote the state with $k = p_\sigma = 0$ by $|\chi\rangle$. Then employ a discrete Fourier transformation,

$$|\tilde{\chi}(\tau)\rangle = (H \otimes H \otimes H) |\chi(\tau)\rangle. \quad (5.68)$$

Now one can check that the amplitudes of $\tilde{\chi}$ can be related to the derivatives of the moduli as [32]

$$\tilde{\chi}_{000} = \frac{i}{2} \frac{\dot{x}_0}{y_0}, \quad \tilde{\chi}_{110} = \frac{i}{2} \frac{\dot{x}_1}{y_1}, \quad \tilde{\chi}_{101} = \frac{i}{2} \frac{\dot{x}_2}{y_2}, \quad \tilde{\chi}_{011} = \frac{i}{2} \frac{\dot{x}_3}{y_3}, \quad (5.69)$$

$$\tilde{\chi}_{111} = \frac{1}{4} \left(\frac{\dot{y}_0}{y_0} - \frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - \frac{\dot{y}_3}{y_3} \right), \quad \tilde{\chi}_{001} = \frac{1}{4} \left(-\frac{\dot{y}_0}{y_0} + \frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - \frac{\dot{y}_3}{y_3} \right) \quad (5.70)$$

where the remaining amplitudes are given by a cyclic shift of the $+$ sign. For static solutions we have vanishing NUT charge and $x_0 = 0$ hence the first of these equations reads $\tilde{\chi}_{000} = 0$. Writing out explicitly the amplitudes $\tilde{\chi}_{jkl}$ in terms of the moduli, warp factor and the charges this constraint implies $p^0 = 0$. Hence our candidate for a non-BPS solution should have only seven nonvanishing Fourier amplitudes and vanishing p^0 charge.

Let us now introduce the notation

$$y_0 = e^{\phi_0}, \quad y_j = e^{\phi_j}, \quad \beta \equiv U - \frac{1}{2}(\phi_1 + \phi_2 + \phi_3), \quad \alpha_j \equiv U + \frac{1}{2}\phi_j \quad (5.71)$$

with and $j = 1, 2, 3$. Now our equations take the form

$$\tilde{\chi}_{111} = \frac{1}{2} \dot{\beta}, \quad \tilde{\chi}_{110} = \frac{i}{2} e^{-\phi_1} \dot{x}_1, \quad \tilde{\chi}_{101} = \frac{i}{2} e^{-\phi_2} \dot{x}_2, \quad \tilde{\chi}_{011} = \frac{i}{2} e^{-\phi_3} \dot{x}_3 \quad (5.72)$$

$$\tilde{\chi}_{001} = \frac{1}{2}(\dot{\alpha}_1 - \dot{\alpha}_2 - \dot{\alpha}_3), \quad \tilde{\chi}_{010} = \frac{1}{2}(\dot{\alpha}_2 - \dot{\alpha}_3 - \dot{\alpha}_1), \quad \tilde{\chi}_{100} = \frac{1}{2}(\dot{\alpha}_3 - \dot{\alpha}_1 - \dot{\alpha}_2). \quad (5.73)$$

With the further charge constraints $q_j = 0$, $q_0 < 0$, and $p^1, p^2, p^3 > 0$ one can see that the equations above are precisely the ones found in the paper of Gimon et.al. [115] characterizing the seed solutions for the so-called $D0 - D4$ system. We remark in closing that one can also verify by an explicit calculation that all of the four algebraically independent four-qubit invariants are again vanishing. This means that the corresponding matrix Q of conserved charges is again nilpotent.

Let us give a brief summary of our results. The central object of our considerations was the *complex* 4-qubit state $|\Lambda\rangle$, satisfying a reality condition. The amplitudes of this state of *odd parity* contain the right invariant one-forms e_α , $\alpha = 0, 1, 2, 3$. On the other hand the 8 amplitudes of *even parity* are related to the 8 amplitudes of a 3-qubit state $|\hat{\chi}\rangle$. We have shown that the state $|\Lambda\rangle$ is connected to the line element on \mathcal{M}_3 . We also realized that this expression for the line element is just the quadratic 4-qubit $SL(2, \mathbb{C})$ invariant. After expressing the 8 amplitudes of the embedded 3-qubit state in terms of the conserved electric, magnetic and NUT charges this invariant also has the physical interpretation as the *extremality* parameter.

Note that one of the qubits of the state $|\Lambda\rangle$ was special. The separability properties of this special qubit are related to the solution being BPS or non-BPS. We demonstrated within our formalism that static, extremal BPS and non-BPS-solutions with vanishing central charge correspond to states for which one of the qubits is separable from the rest. On the other hand using the non-BPS seed solution for nonvanishing central charge we have shown that $|\Lambda\rangle$ in this case is entangled. We revealed a connection between the classification of *nilpotent states* within the realm of quantum information theory and the similar classification of nilpotent orbits. The details of this connection will be explored further in the next section.

5.5. Four-qubit entanglement from string theory

In the proceeding section it was shown how the time-like reduced STU model may be naturally related to a four-qubit system, rephrasing various important features of the black hole solutions in quantum information theoretic terms. In particular, certain BPS and non-BPS spherically symmetric black hole solutions were related to (partially and totally) entangled four-qubit states. Here, developing this correspondence, we will describe how the classification of all extremal, both single-centre and multi-centre, black hole solutions provides a complete characterisation of the four-qubit entanglement classes.

The extremal black hole solutions are determined by the nilpotent orbits of the 3-dimensional U-duality group [104, 108, 109, 116, 117]. The nilpotent orbits are then related to the four-qubit entanglement classes through the Kostant-Sekiguchi correspondence [22, 23, 32, 118, 119]. Using these tools we find that there are 31 four-qubit entanglement families, which reduce to 9 under permutations of Alice, Bob, Charlie and Dave in agreement with the quantum information and mathematical literature [120, 121].

An interesting new feature, first treated in [122, 123], of the four-qubit correspondence, which goes beyond the three-qubit case, is the appearance of interacting multi-centre black hole solutions as will be described below.

Although two and three qubit entanglement is well-understood (see e.g. [14]), the literature on four qubits can be confusing and seemingly contradictory, as illustrated in Table I. This is due in part to genuine calculational disagreements, and in part to the use of distinct (but in principle consistent and complementary) perspectives on the criteria for classification.

TABLE I: Various results on four-qubit entanglement.

Paradigm	Author	Year	Ref	result mod perms		result incl. perms	
classes	Wallach	2004	[124]	?		90	
	Lamata et al	2006	[125]	8 genuine,	5 degenerate	16 genuine,	18 degenerate
	Cao et al	2007	[126]	8 genuine,	4 degenerate	8 genuine,	15 degenerate
	Li et al	2007	[127]	?		≥ 31 genuine,	18 degenerate
	Akhtarshenas et al	2010	[128]	?		11 genuine,	6 degenerate
	Buniy et al	2010	[129]	21 genuine,	5 degenerate	64 genuine,	18 degenerate
families	Verstraete et al	2002	[120]	9		?	
	Chterental et al	2007	[121]	9		?	
	String theory	2010	[22]	9		31	

On the one hand, there is the “covariant” approach which distinguishes the SLOCC orbits by the vanishing or not of $[SL(2, \mathbb{C})]^{\otimes n}$ covariants/invariants. This philosophy is adopted for the 3-qubit case in [14, 33]. The analogous 4-qubit case was treated, with partial results, in [130].

On the other hand, there is the “normal form” approach which considers “families” of orbits. An arbitrary state may be transformed into one of a finite number of normal forms. If the normal form depends on some of the algebraically independent SLOCC invariants it constitutes a family of orbits parametrised by these invariants. On the other hand, a parameter-independent family contains a single orbit. This philosophy is adopted for the 4-qubit case in [120, 121]. There are four algebraically independent SLOCC invariants [130]. Up to permutation of the four qubits, these authors found 6 parameter-dependent families called G_{abcd} , L_{abc_2} , $L_{a_2b_2}$, $L_{a_20_3\oplus\bar{1}}$, L_{ab_3} , L_{a_4} and 3 parameter-independent families called $L_{0_3\oplus\bar{1}0_3\oplus\bar{1}}$, $L_{0_5\oplus\bar{3}}$, $L_{0_7\oplus\bar{1}}$, see Table II.

TABLE II: The 9 ways of entangling four qubits

Verstraete/Chterental Family	Representative state
G_{abcd}	$\frac{a+d}{2}(0000\rangle + 1111\rangle) + \frac{a-d}{2}(0011\rangle + 0011\rangle) + \frac{b+c}{2}(0101\rangle + 1010\rangle) + \frac{b-c}{2}(0110\rangle + 1001\rangle)$
L_{abc_2}	$\frac{a+b}{2}(0000\rangle + 1111\rangle) + \frac{a-b}{2}(0011\rangle + 0011\rangle) + c(0101\rangle + 1010\rangle) + 0110\rangle$
$L_{a_2b_2}$	$a(0000\rangle + 1111\rangle) + b(0101\rangle + 1010\rangle) + 0110\rangle + 0011\rangle$
L_{ab_3}	$a(0000\rangle + 1111\rangle) + \frac{a+b}{2}(0101\rangle + 1010\rangle) + \frac{a-b}{2}(0110\rangle + 1001\rangle) + \frac{i}{\sqrt{2}}(0001\rangle + 0010\rangle + 0111\rangle + 1011\rangle)$
L_{a_4}	$a(0000\rangle + 0101\rangle + 1010\rangle + 1111\rangle) + i 0001\rangle + 0110\rangle - i 1011\rangle$
$L_{a_2 0_3 \oplus \bar{1}}$	$a(0000\rangle + 1111\rangle) + 0011\rangle + 0101\rangle + 0110\rangle$
$L_{0_5 \oplus \bar{3}}$	$ 0000\rangle + 0101\rangle + 1000\rangle + 1110\rangle$
$L_{0_7 \oplus \bar{1}}$	$ 0000\rangle + 1011\rangle + 1101\rangle + 1110\rangle$
$L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$	$ 0000\rangle + 0111\rangle$

To illustrate the difference between these two approaches, consider the separable EPR-EPR state $(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$. Since this is obtained by setting $b = c = d = 0$ in Table II, it belongs to the G_{abcd} family, whereas in the covariant approach it forms its own class. Similarly, a totally separable A - B - C - D state, such as $|0000\rangle$, for which all covariants/invariants vanish, belongs to the family L_{abc_2} , which however also contains genuine four-way entangled states. These interpretational differences were also noted in [125]. As we shall see, our black hole perspective lends itself naturally to the “normal form” framework.

In order to relate the extremal BH solutions to the entanglement classes of four qubits, we invoke the aforementioned Kostant-Sekiguchi theorem [118, 119]. By applying the Kostant-Sekiguchi correspondence to the Cartan decomposition,

$$\mathfrak{so}(4, 4) = 4\mathfrak{sl}(2) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}) \quad (5.74)$$

one can state that the nilpotent orbits of $\mathrm{SO}_0(4, 4)$ acting on its adjoint representation, which classify the extremal black hole solutions, are in one-to-one correspondence with the nilpotent orbits of $[\mathrm{SL}(2, \mathbb{C})]^4$ acting on its fundamental $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$ representation and, hence, with the classification of 4-qubit entanglement. Note furthermore that it is the complex qubits that appear automatically, thereby relaxing the restriction to real qubits (sometimes called rebits) that featured in earlier versions of the BH/qubit correspondence.

It follows that there are 31 nilpotent orbits for four qubits under SLOCC [22]. For each nilpotent orbit there is precisely one family of SLOCC orbits since each family contains one nilpotent orbit on setting all invariants to zero. The nilpotent orbits and their associated families are summarized in Table VIII [22], which is split into upper and lower sections according as the nilpotent orbits belong to parameter-dependent or parameter-independent families.

5.5.1. Extremal black hole solutions and nilpotent orbits

In section 5.5.3 we saw how one can study the stationary black hole solutions of the STU model by performing a time-like dimensional reduction. The intuition is that since a strictly stationary spacetime by definition admits an everywhere time-like Killing vector it can be dimensionally reduced to a 3-dimensional field theory. Solutions of the 3-dimensional theory then up-lift to stationary solutions of its 4-dimensional parent theory, providing the ideal framework for the study of stationary 4-dimensional black holes, as first demonstrated to great effect in [102].

For 4-dimensional supergravity theories with scalars ϕ living in a symmetric space $\mathcal{M}_4 = G_4/H_4$, where G_4 is the U-duality group and H_4 its maximal compact subgroup we can exploit these group-theoretic structures to systematically characterise the various classes of stationary black holes. Note, this includes all supergravity theories with $\mathcal{N} > 2$ and a large class of $\mathcal{N} = 2$ theories, including the Einstein-Maxwell theories such as the STU model.

Performing a time-like reduction of such models leads to a theory of 3-dimensional gravity coupled to the 3-dimensional scalars Φ through a nonlinear sigma model described by the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\sqrt{h}R[h] + g_{mn}\partial^a\Phi^m\partial^b\Phi^n h_{ab}. \quad (5.75)$$

The scalars Φ come not only directly from the 4-dimensional scalars, but also the metric and the 3-dimensional gauge fields after dualization, which in turn originate from both the metric and the 4-dimensional gauge fields. This procedure is described in detail for the *STU* model in section 5.5.3. Einstein's equations and equations of motion for the scalar fields derived from (5.75) are given respectively by,

$$R_{ab} - \frac{1}{2}h_{ab}R = g_{mn}\partial_a\Phi^m\partial_b\Phi^n - \frac{1}{2}h_{ab}g_{mn}\partial_c\Phi^m\partial^c\Phi^n \quad (5.76)$$

and

$$\nabla_a\nabla^a\Phi^m + \Gamma_{np}^m\partial_a\Phi^n\partial^a\Phi^p = 0. \quad (5.77)$$

The 3-dimensional scalars Φ parametrize a pseudo-Riemannian symmetric space $\mathcal{M}_3 = G_3/H_3^*$, with line element given by the scalar manifold metric g_{mn} . Here, G_3 is the 3-dimensional U-duality group and H_3^* is the maximally non-compact real form of $H_3(\mathbb{C})$, the complexification of the maximal compact subgroup $H_3 \subset G_3$. If we had perform instead a space-like reduction we would have obtained the Riemannian symmetric space $\mathcal{M}_3 = G_3/H_3$. Such a model may be described in terms of a coset representative $\mathcal{V} \in G_3/H_3^*$,

$$\mathcal{V} \mapsto g\mathcal{V}, \quad g \in G_3, \quad \mathcal{V} \mapsto \mathcal{V}h(\Phi), \quad h \in H_3^*, \quad (5.78)$$

containing all the scalar degrees of freedom, as well as the three-dimensional metric h_{ab} , which carries no physical degrees of freedom in three dimensions.

The Maurer-Cartan form $\mathcal{V}^{-1}d\mathcal{V}$ decomposes as,

$$\mathcal{V}^{-1}d\mathcal{V} = P + B, \quad P = P_a dx^a \in \mathfrak{p}, \quad B = B_a dx^a \in \mathfrak{h}_3^*, \quad (5.79)$$

where

$$\mathfrak{g}_3 = \mathfrak{h}_3^* \oplus \mathfrak{p}, \quad (5.80)$$

and $\mathfrak{g}_3, \mathfrak{h}_3^*$ denote the Lie algebras of G_3, H_3^* , respectively.

In this language the Einstein equation and the scalar equations of motion are respectively given by,

$$R_{ab} - \frac{1}{2}h_{ab}R = \text{Tr } P_a P_b, \quad d \star P + [B, \star P] = 0 \quad (5.81)$$

and the Bianchi identity gives

$$dB + B^2 = -P^2, \quad dP + [B, P] = 0. \quad (5.82)$$

A few lines using the above shows

$$d \star \mathcal{V} P \mathcal{V}^{-1} = 0, \quad (5.83)$$

so we have a \mathfrak{g}_3 valued Noether current $\star \mathcal{V} P \mathcal{V}^{-1}$, which for a 2-cycle Σ defines a \mathfrak{g}_3 valued Noether charge matrix,

$$Q = \frac{1}{4\pi} \int_{\Sigma} \star \mathcal{V} P \mathcal{V}^{-1}. \quad (5.84)$$

Restricting our attention to weakly extremal solutions, i.e. assuming the 3-dimensional spatial slice to be flat, from (5.76) we have,

$$g_{mn}\partial_a\Phi^m\partial_b\Phi^n = 0, \quad (5.85)$$

or equivalently,

$$\text{Tr } P_a P_b = 0, \quad (5.86)$$

so the moduli decouple from gravity. With these restrictions the solutions correspond to harmonics maps from $\mathbb{R}^3/\{x_A\}$ to \mathcal{M}_3 , where the x_A , $A = 1, \dots, k$ correspond to a finite set of removed points.

Further imposing spherical symmetry, focusing on single-centre solutions, the scalars will depend only on the radial coordinate and solutions to (5.85) are given by null geodesics in \mathcal{M}_3 [102]. In this case the Noether charge $Q \in \mathfrak{g}_3$ given in (5.84) will be *nilpotent*. In general, an element $X \in \text{End}(V)$, where V is a finite dimensional complex vector space, is said to be *semisimple* if its eigenvectors form a basis for V , in which case it is diagonalizable. An element X is said to be nilpotent if $X^n = 0$ for some finite n . This definition naturally applies to the adjoint representation of Lie algebras,

$$\text{ad} : \mathfrak{g} \mapsto \text{End}(\mathfrak{g}); \quad X \mapsto \text{ad}_X \quad \text{where} \quad \text{ad}_X Y := [X, Y]. \quad (5.87)$$

An element $X \in \mathfrak{g}$ is semisimple (resp. nilpotent) if ad_X is a semisimple (resp. nilpotent) endomorphism. Then the Jordan Decomposition theorem says for any $X \in \text{End}(V)$ there exists a unique commuting pair $X_s, X_n \in \text{End}(V)$ such that $X = X_s + X_n$, where X_s is semisimple and X_n is nilpotent.

A solution corresponding to a given null geodesic is precisely determined by the nilpotent conserved charge Q [108, 109, 116]. Since $\mathcal{V} \mapsto g\mathcal{V}h$ and $P \mapsto h^{-1}Ph$ under U-duality, the Noether charge transforms according as $Q \mapsto gQg^{-1}$. Consequently, the physically distinct weakly extremal black hole solutions are classified in terms of the adjoint orbits,

$$\mathcal{O}_Q = G_3 \cdot Q := \{gQg^{-1} \mid g \in G_3\}, \quad (5.88)$$

for Q nilpotent [109, 116]. We will refer to these simply as the *nilpotent orbits of G_3* . Moreover the scalar momentum P will lie in the intersection of \mathcal{O}_Q with the Lie algebra component \mathfrak{p} .

To summarise, the single-centre extremal black hole solutions are in one-to-one correspondence with the nilpotent orbits of G_3 . Well not quite, one must still pay due diligence to the regularity of the solutions. A weakly extremal solution is not necessarily a smooth extremal solution. In particular, it was recognised in [131] that regularity places a constraint on the degree n of nilpotency of Q , i.e. the smallest n such that $[\text{ad}_Q]^n = 0$. This is a subtle matter in general, but for our case of interest, the *STU* model, it turns out we require $Q^3 = 0$ [109], where Q lies strictly in \mathfrak{p} , which is the case for asymptotically flat solutions since \mathcal{V} goes to identity in the asymptotic limit.

Dropping the assumption of spherical symmetry we return to the case in which the stationary solutions correspond to harmonics maps from $\mathbb{R}^3/\{x_A\}$ to \mathcal{M}_3 and the scalars depend on generically on \mathbb{R}^3 . In order to study the extremal multi-centre solutions it was assumed in [117] that the cycles Σ defining the Noether charges are characterised by the black hole centres x_A they enclose. In particular, the Noether charge Q_A associated to a cycle enclosing a single centre at x_A is characterised by the pole of $\star\mathcal{V}P\mathcal{V}^{-1}$ at $x = x_A$. Since the presence of other centres modifies the horizon geometry of a given centre by subleading corrections, regularity requires the associated Noether charge to satisfy the single-centre constraint [117], that is $Q_A^3 = 0$ for all A .

However, the crucial observation made in [117] is that, while the individual centres must satisfy $Q_A^3 = 0$, the Noether charge Q of a cycle Σ_I enclosing a set of centres x_A , $A \in I$, which is given by,

$$Q = \sum_A Q_A, \quad (5.89)$$

need not be of nilpotency degree 3. It is then reasonable to anticipate that the nilpotent orbits which have nilpotency degree too high to accommodate regular single-centre black holes do support *regular multi-centre solutions*, where each centre individually falls into one of the well-defined single-centre orbits. For the *STU* model this is explicitly shown to be the case in [117] and has recently been extended to $\mathcal{N} = 8$ supergravity in [132]. We will briefly describe some the features of the *STU* solutions in the following sections.

5.5.2. Nilpotent orbits of $\text{SO}(4, 4)$

As described in section 5.5.3 the *STU* model has $\mathcal{M}_3 = \text{SO}(4, 4)/\text{SL}(2, \mathbb{R})^{\otimes 4}$. Here we summarise the structure of the nilpotent orbits of the identity component $\text{SO}_0(4, 4)$.

The classification of nilpotent orbits is a rather elegant and well developed subject. We will not describe here the general theory of the orbit classification, but interested reader can refer to [119, 133]. We will merely present the results and their labelling for the relevant case of $\text{O}(p, q)$, where $p + q = n$.

The nilpotent orbits of $\text{O}(p, q)$ may be labelled by “signed Young tableaux” (sometimes referred to as *ab*-diagrams in the mathematical literature). A signed Young tableau for $\text{O}(p, q)$ is an n box Young tableau whose boxes filled with signs $+/-$ such that:

1. The signs alternate along the rows.

2. The total number of +’s must be p .
3. Rows of even length must come in pairs such that if one row starts with “+” the other row starts with “−”.

Two such diagrams are equivalent if they may be related by row permutation. There is precisely one nilpotent orbit of $O(p, q)$ on $\mathfrak{so}(p, q)$ for each equivalence class of signed Young tableaux.

Since the identity component $SO_0(p, q)$ has index 4 in $O(p, q)$, for each nilpotent $O(p, q)$ orbit there may be either 1, 2 or 4 nilpotent $SO_0(p, q)$ orbits. This number is also determined by the corresponding signed Young tableau. If the middle sign of every odd length row is “−” (“+”) there are 2 orbits and we label the diagram to its left (right) with a I or a II . If it only has even length rows there are 4 orbits and we label the diagram to both its left and right with a I or a II . If it is none of these it is said to be stable and there is only one orbit.

Following these rules for our case of $SO_0(4, 4)$ we find 31 labelled signed Young tableaux as given in Table VIII. The corresponding to the 31 nilpotent orbits are presented in the same table. The closure ordering of the orbits is given in Figure 8.

The Kostant-Sekiguchi correspondence [118, 119] then implies that the nilpotent orbits of $SO_0(4, 4)$ acting on the adjoint representation **28** are in one-to-one correspondence with the nilpotent orbits of $[SL(2, \mathbb{C})]^4$ acting on the fundamental representation $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$ and hence with the classification of four-qubit entanglement [192].

5.5.3. Extremal black holes and entanglement classes.

We are now in a position to summarise the single-centre/multi-centre black hole solutions of the STU model and their corresponding four-qubit entanglement classes. The details of the orbit classification and the black hole solutions may be found in the original references [70, 104, 108, 109, 116, 117, 119, 134–139]. The explicit mappings to the four-qubit entanglement classes via the Kostant-Sekiguchi correspondence may be found in [122]. The 31 classes reduce to 9 under the permutation of the qubits, as described in [23]. These 9 classes of orbits split into two: those for which the orbits have dimension less than 20, which admit regular single-centre solutions, and those for which the orbits have dimension 20 or greater, which only admit regular multi-centre solutions. The basic structure of the orbit classification is presented in Figure 7.

The trivial nilpotent orbit corresponds to the family of purely semi-simple states G_{abcd} , which are identically zero for all the four-qubit invariants set to zero. Equally, the non-trivial black hole solutions in this orbit are by definition non-extremal and are not treated here.

The supergravity interpretation of the $SO_0(4, 4)$ -nilpotent orbits of $\dim_{\mathbb{R}} \leq 18$ (corresponding to single-centre solutions) considered below is based on the embedding of the STU model in $\mathcal{N} = 8, D = 4$ supergravity discussed *e.g.* in [79]. This amounts to identifying the $\mathcal{N} = 2$ central charge, introduced in (5.55), and the three associated “matter” charges with the four skew-eigenvalues Z_i ($i = 1, \dots, 4$ throughout) of the $\mathcal{N} = 8$ central charge matrix as follows [79, 140]

$$Z \equiv Z_1; \sqrt{g^{s\bar{s}}}\overline{D_s}Z \equiv iZ_2; \sqrt{g^{t\bar{t}}}\overline{D_{\bar{t}}}Z \equiv iZ_3; \sqrt{g^{u\bar{u}}}\overline{D_{\bar{u}}}Z \equiv iZ_4. \quad (5.90)$$

Thus, the effective BH potential V_{BH} , its criticality conditions or, equivalently, the attractor equations, and the quartic invariant $I_4 = -D(\psi)$ of $\mathcal{N} = 2, D = 4$ STU model can be traded for the ones pertaining to maximal supergravity, respectively reading [77, 79, 141]:

$$V_{BH} = \sum_i |Z_i|^4; \quad (5.91)$$

$$\partial_\phi V_{BH} = 0 \Leftrightarrow Z_i Z_j + \overline{Z_k} \overline{Z_l} = 0, \forall i \neq j \neq k \neq l; \quad (5.92)$$

$$I_4 = \sum_i |Z_i|^4 - 2 \sum_{i < j} |Z_i|^2 |Z_j|^2 + 4 \left(\prod_i Z_i + \prod_i \overline{Z_i} \right). \quad (5.93)$$

Note, the central charge $Z_1 = Z$ is on a different footing to the matter charges Z_2, Z_3, Z_4 for the STU model, while they are all equivalent from the $\mathcal{N} = 8$ perspective.

Here we summarise the black hole solutions and their associated entanglement classes. They are grouped into the three broad classes of small, large and multi-centre. Each orbit is labelled $\mathcal{O}_{\dim_{\mathbb{R}}}$, where $\dim_{\mathbb{R}}$ is the real dimension of the orbit.

Small black holes and totally/partially separable four-qubit states.

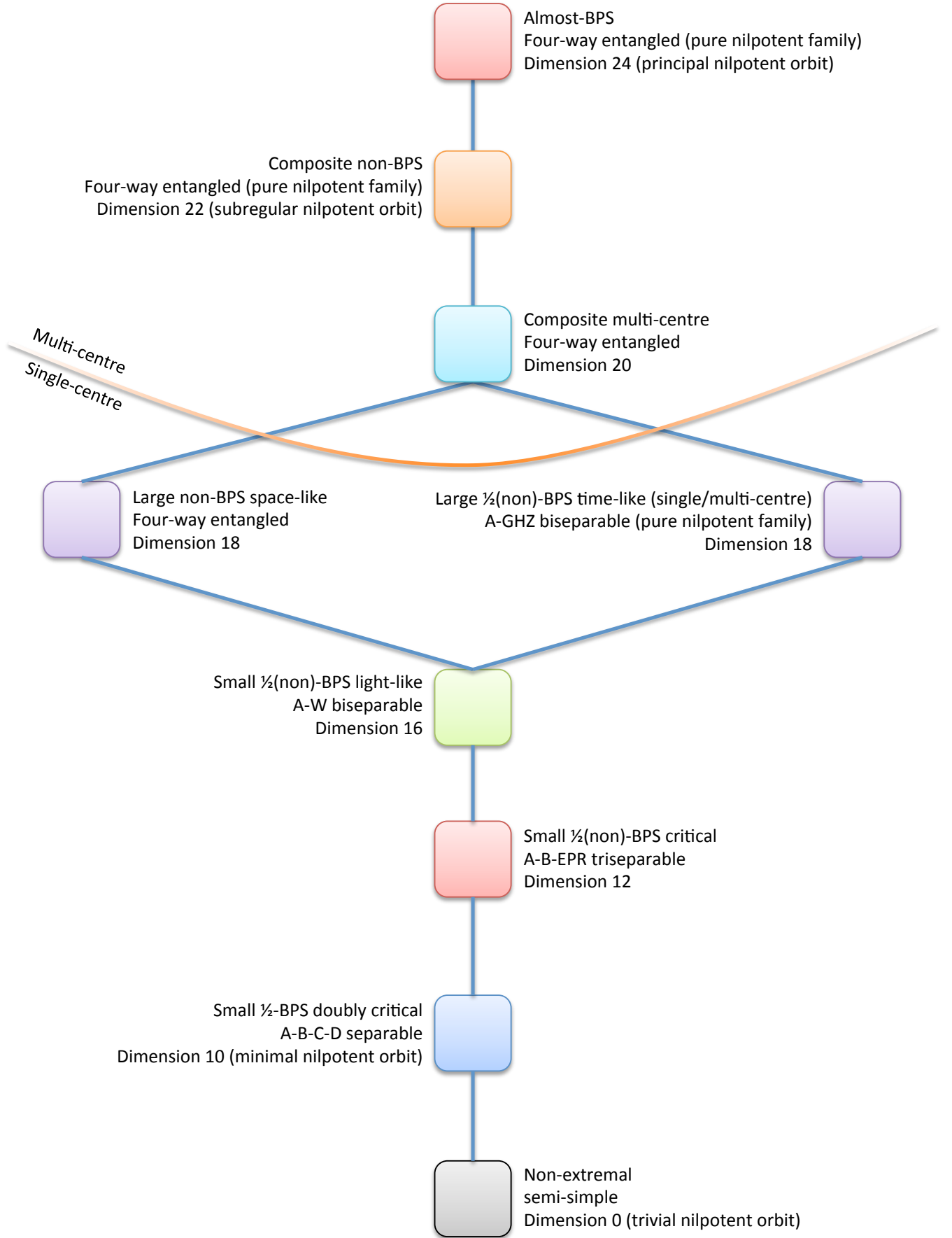


FIG. 7: Basic structure of the orbit classification with orbit dimension, black hole class and entanglement properties. Above the horizontal line the orbits only admit regular multi-centre solutions [117].

a. \mathcal{O}_{10} : A - B - C - D and doubly-critical $\frac{1}{2}$ -BPS black holes. There is a unique orbit in this class, given by the unique unlabelled signed Young tableaux

$$\begin{array}{|c|c|} \hline + & - \\ \hline - & + \\ \hline + & \\ \hline - & \\ \hline + & \\ \hline - & \\ \hline \end{array} \quad (5.94)$$

The doubly-critical $\frac{1}{2}$ -BPS orbit comprises of small single-centre single-charge black holes with vanishing classical entropy [89]. The scalar fields take constant values along the attractor flow. The uniqueness of the doubly-critical $\frac{1}{2}$ -BPS orbit was explained in [138]. The permutation invariance and uniqueness may be understood in terms of the conditions on dressed charges defining the solution [23],

$$\begin{cases} |Z_1|^2 = |Z_2|^2 = |Z_3|^2 = |Z_4|^2; \\ Z_i Z_j - \overline{Z_k Z_l} = 0, \forall i \neq j \neq k \neq l, \end{cases} \quad (5.95)$$

which are themselves permutation invariant.

The corresponding four-qubit entanglement class given by Kostant-Sekiguchi is the totally separable A - B - C - D states with permutation invariant representative state,

$$|0000\rangle. \quad (5.96)$$

This class belongs to the entanglement family L_{abc_2} . The uniqueness of the Young tableau is reflected in the invariance of the A - B - C - D class under the four-qubit permutation group.

b. \mathcal{O}_{12} : A - B -EPR and critical $\frac{1}{2}$ -BPS/non-BPS black holes. This class has six distinct $\text{SO}_0(4, 4)$ -orbits, given by the six (labelled) signed Young tableaux,

$$\begin{array}{|c|c|c|} \hline - & + & - \\ \hline + & & \\ \hline - & & \\ \hline + & & \\ \hline - & & \\ \hline + & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline + & - & + \\ \hline - & & \\ \hline + & & \\ \hline - & & \\ \hline + & & \\ \hline - & & \\ \hline \end{array}, \quad \left(I, II \quad \begin{array}{|c|c|} \hline + & - \\ \hline - & + \\ \hline + & - \\ \hline - & + \\ \hline \end{array} \quad I, II \right). \quad (5.97)$$

This class gives the critical $\frac{1}{2}$ -BPS/non-BPS orbits which comprises of small single-centre two-charge black holes with vanishing classical entropy. This set of solutions may be defined in terms of an $[\text{SL}(2, \mathbb{R})]^3$ -invariant set of constraints [23, 89, 142]:

$$2Z_i \overline{Z_i}^2 - 2\overline{Z_i} \sum_{j \neq i} |Z_j|^2 + 4 \prod_{j \neq i} Z_j = 0, \quad i = 1, 2, 3, 4. \quad (5.98)$$

This set of constraints is manifestly invariant under cyclic permutations of the index $j \neq i$.

The four constraints (5.98) admit six representative solutions [23]:

$$\begin{cases} \text{I:} & Z_1 = 0 = Z_2, \quad Z_3 = Z_4 \neq 0; \\ \text{II:} & Z_1 = 0 = Z_3, \quad Z_2 = Z_4 \neq 0; \\ \text{III:} & Z_1 = 0 = Z_4, \quad Z_2 = Z_3 \neq 0; \\ \text{IV:} & Z_2 = 0 = Z_3, \quad Z_1 = Z_4 \neq 0; \\ \text{V:} & Z_2 = 0 = Z_4, \quad Z_1 = Z_3 \neq 0; \\ \text{VI:} & Z_3 = 0 = Z_4, \quad Z_1 = Z_2 \neq 0, \end{cases} \quad (5.99)$$

These split into three non-BPS solutions $\{\text{I}, \text{II}, \text{III}\}$ with $Z_1 = 0$ and three 1/2-BPS solutions $\{\text{IV}, \text{V}, \text{VI}\}$ with $Z_1 \neq 0$. Each of the six representative solutions are related under the permutations. To understand the relationship with the signed tableaux in (5.97) we can define the sets:

$$\{Z_i = Z_{i+1} = 0, Z_{i+2} = Z_{i+3} \neq 0\}; \quad \{Z_i = Z_{i+2} = 0, Z_{i+1} = Z_{i+3} \neq 0\}, \quad (5.100)$$

which correspond to $\{\text{I}, \text{III}, \text{IV}, \text{VI}\}$ and $\{\text{II}, \text{V}\}$ respectively. These two sets are given by the two distinct types of Young tableaux. Embedding the STU model in the $\mathcal{N} = 8$ the four Z_i are all put on the same footing and each orbit is identified, yielding a single class of critical two-charge 1/4-BPS black holes.

The corresponding four-qubit entanglement classes given by Kostant-Sekiguchi are the six possible A - B -EPR type configurations, where one pair of qubits forms a Bell state while the remaining two are separable,

$$|00\rangle \otimes (|00\rangle + |11\rangle). \quad (5.101)$$

There are six such classes corresponding to the six possible entangled pairs. These classes belong to the entanglement families of type $L_{a_2b_2}$. Clearly these six classes collapse into a single class under the four-qubit permutations.

c. \mathcal{O}_{16} : A - W and light-like $\frac{1}{2}$ -BPS/non-BPS black holes. This class has four distinct $\text{SO}_0(4, 4)$ -orbits, given by the four (labelled) signed Young tableaux,

$$\left(I, II \begin{array}{|c|c|c|} \hline + & - & + \\ \hline - & + & \\ \hline + & - & \\ \hline - & & \\ \hline \end{array} \right) \quad \left(\begin{array}{|c|c|c|} \hline - & + & - \\ \hline + & - & \\ \hline - & + & \\ \hline + & & \\ \hline \end{array} I, II \right). \quad (5.102)$$

This class gives the light-like $\frac{1}{2}$ -BPS/non-BPS orbits which comprises of small single-centre three-charge black holes with vanishing classical entropy. This set of solutions may be defined in terms of an $[\text{SL}(2, \mathbb{R})]^3$ -invariant set of constraints on the central/matter charges [23, 89, 142]:

$$\sum_i |Z_i|^4 - 2 \sum_{i < j} |Z_i|^2 |Z_j|^2 + 4 \left(\prod_i Z_i + \prod_i \overline{Z}_i \right) = 0. \quad (5.103)$$

This set of constraints is manifestly invariant under cyclic permutations of Z_1, Z_2, Z_3, Z_4 .

The constraint (5.103) admits four representative solutions [23], each corresponding to one of the four labelled Young tableaux in (5.102). These split into three non-BPS solutions and one 1/2-BPS solution, according as the particular properties of the central/matter charges. See [23] for details. Each of the four representative solutions are related under the permutations. Again, embedding the STU model in the $\mathcal{N} = 8$ theory the four Z_i are all put on the same footing and each orbit is identified, yielding a single class of light-like three-charge 1/8-BPS black holes.

The corresponding four-qubit entanglement classes given by Kostant-Sekiguchi are the four possible A - W type configurations, where three of the qubits are in the totally entangled W state while the remaining one is separable,

$$|0\rangle \otimes (|001\rangle + |010\rangle + |100\rangle). \quad (5.104)$$

There are obviously four such classes corresponding to the four possible choices of separable qubit. These classes belong to the entanglement families of type $L_{a_20_3\oplus\overline{1}}$. These four classes collapse into a single class under the four-qubit permutations.

Large black holes and three-way/totally entangled four-qubit states.

d. \mathcal{O}_{18a} : A -GHZ and time-like $\frac{1}{2}$ -BPS/non-BPS black holes. This class has four distinct $\text{SO}_0(4, 4)$ -orbits, given by the four (labelled) signed Young tableaux,

$$\left(I, II \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & - & + \\ \hline - & & \\ \hline - & & \\ \hline \end{array} \right) \quad \left(\begin{array}{|c|c|c|} \hline - & + & - \\ \hline - & + & - \\ \hline + & & \\ \hline + & & \\ \hline \end{array} I, II \right). \quad (5.105)$$

This class gives the time-like $\frac{1}{2}$ -BPS/non-BPS orbits which comprises of the large single-centre four-charge black holes with non-zero classical entropy [23, 89, 134, 142],

$$S = \pi \sqrt{-D(\psi)}, \quad (5.106)$$

where ψ is the three-qubit state built out of the 4-dimensional electromagnetic charges as in (2.7). Here, Cayley's hyperdeterminant is constrained to be negative $D(\psi) < 0$, or equivalently $\mathcal{I}_4 > 0$.

The four orbits correspond to the four classes of representative solution to the constraints $Z_i Z_j + \overline{Z}_k \overline{Z}_l = 0, \forall i \neq j \neq k \neq l$ and $\mathcal{I}_4 > 0$ given by [23, 89, 134, 142],

$$Z_i \neq 0, Z_j = Z_k = Z_l = 0, \quad \forall i \neq j \neq k \neq l. \quad (5.107)$$

As explained in [104, 134, 138] there just is one 1/2-BPS $\text{SO}_0(4, 4)$ -orbit given by the case $Z_1 \neq 0$, which is on a different footing from the STU perspective to the remaining three possibilities. These are indeed all non-BPS, but actually fall into two separate classes as explained in [23, 134].

Embedding these solutions in the $\mathcal{N} = 8$ theory they become equivalent and correspond to a single class of 1/8-BPS large black holes.

This class also includes the interacting 1/2-BPS multi-centre solutions first discovered by Bates and Denef [117, 143, 144], where every centre is individually 1/2-BPS. Accordingly, this orbit is referred to as the “Denef” system in [117].

The corresponding four-qubit entanglement classes given by Kostant-Sekiguchi are the four possible A -GHZ type configurations, where three of the qubits are in the totally entangled GHZ state, with non-zero three-tangle $\tau_{ABC} \neq 0$, while the remaining one is separable,

$$|0\rangle \otimes (-|111\rangle + |001\rangle + |010\rangle + |100\rangle). \quad (5.108)$$

There are obviously four such classes corresponding to the four possible choices of separable qubit. These classes belong to the entanglement families of type $L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$. This is a pure single nilpotent orbit in the sense that all four algebraically independent SLOCC invariants are identically zero for the normal form. These four classes collapse into a single class under the four-qubit permutations.

e. \mathcal{O}_{18b} : Genuine four-way entanglement and space-like non-BPS black holes. There is a unique orbit in this class, given by the unique unlabelled signed Young tableaux

$$\begin{array}{|c|c|c|} \hline - & + & - \\ \hline + & - & + \\ \hline - & & \\ \hline + & & \\ \hline \end{array} \quad (5.109)$$

This class gives the space-like non-BPS solution which comprises of the large single-centre four-charge black holes with non-zero classical entropy [23, 89, 134, 142],

$$S = \pi \sqrt{D(\psi)}, \quad (5.110)$$

where Cayley’s hyperdeterminant is now constrained to be positive $D(\psi) > 0$, or equivalently $\mathcal{I}_4 < 0$.

This solution is determined by the representative solution to the constraints $Z_i Z_j + \bar{Z}_k \bar{Z}_l = 0$ and $\mathcal{I}_4 < 0$ given by [23, 89, 134, 142],

$$Z_i = \rho e^{i\frac{\varphi}{4}}, \quad \rho \in \mathbb{R}_0^+, \quad \forall i, \quad \varphi = \pi + 2k\pi, \quad k \in \mathbb{Z}, \quad (5.111)$$

which is permutation invariant as required for consistency with our single Young tableau. Accordingly, when embedded in the $\mathcal{N} = 8$ theory this solution continues to be non-BPS. However, unlike the previous class it does not admit any interacting multi-centre solutions [103].

Interestingly, the corresponding four-qubit entanglement class given by Kostant-Sekiguchi is the first instance of genuine four-way entanglement for a nilpotent state. These states belongs to the class L_{ab3} . A representative state is given by,

$$\frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle - |0111\rangle - |1011\rangle). \quad (5.112)$$

While the permutation invariance is not manifest in this case, all states obtained from (5.112) by swapping the qubits are related by $[\text{SL}(2, \mathbb{C})]^4$.

Multi-centre black holes and four-way entangled four-qubit states. The three remaining classes of $\text{SO}_0(4, 4)$ -orbits do not admit regular single centre solutions since they all have representative nilpotent elements of degree greater than three. However, they do admit interacting multi-centre solutions for which, although the total charge has degree greater than three, the individual centres have charges of nilpotency degree three and belong to the orbits described above [117]. Consequently, the three associated nilpotent four-qubit entanglement classes can only be interpreted in terms of interacting multi-centre solutions. What this means for the entanglement properties for these classes remains unclear.

f. \mathcal{O}_{22} : $L_{0_5 \oplus \bar{3}}$ and non-BPS composite black holes. This class has four distinct $\text{SO}_0(4, 4)$ -orbits, given by the four (labelled) signed Young tableaux,

$$\left(I, II \begin{array}{|c|c|c|c|c|} \hline - & + & - & + & - \\ \hline + & - & + & & \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|c|c|c|} \hline + & - & + & - & + \\ \hline - & + & - & & \\ \hline \end{array} I, II \right). \quad (5.113)$$

In the mathematical literature it is referred to as the “subregular” orbit meaning it is has second highest dimension and nilpotency degree.

The two-centre solutions deriving from this orbit were recently obtained in [117]. At the horizon of each centre the scalar momentum P will lie in the single-centre orbit of space-like non-BPS black holes given by the Young tableaux in (5.109). Hence, it can be understood as an interacting composite system where each centre is space-like non-BPS. Note, this does not

contradict the fact that there are no interacting multi-centre space-like non-BPS black holes in the orbit defined by (5.109), since away from the horizons P will lie strictly in the subregular nilpotent orbit considered here and given by (5.113).

This can be understood clearly in terms of a representative nilpotent charge $Q \in \mathfrak{p}$ of the subregular orbit. Q satisfies $Q^5 = 0$ but not $Q^4 = 0$ and hence there are no regular single-centre solutions. However, it is simple to show [117] that there exists a finite set of charges Q_A , satisfying $Q_A^3 = 0$ and, in particular, $Q_A \in \mathcal{O}_{18b}$, such that,

$$Q = \sum_A Q_A, \quad (5.114)$$

for which one can construct an everywhere regular solution. Another interesting feature that we have yet to encounter is that each black hole in the two-centre case can carry intrinsic angular momentum along the axis of interaction.

Turning our attention to the corresponding class of entangled states given by Kostant-Sekiguchi we obtain the first example of a purely nilpotent family with genuine four-way entanglement:

$$|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle. \quad (5.115)$$

g. \mathcal{O}_{24} : $L_{0_{7\oplus\bar{1}}}$ and Almost-BPS composite black holes. This class has four distinct $\mathrm{SO}_0(4,4)$ -orbits, given by the four (labelled) signed Young tableaux,

$$\left(I, II \begin{array}{|c|c|c|c|c|c|} \hline + & - & + & - & + & - \\ \hline - & & & & & \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|c|c|c|c|} \hline - & + & - & + & - & + \\ \hline + & & & & & \\ \hline \end{array} I, II \right). \quad (5.116)$$

In the mathematical literature it is referred to as the “principle” orbit meaning it is has the highest dimension and nilpotency degree.

The black hole solutions originating from this orbit form the “almost-BPS” class, originally obtained by different methods in [135–137]. In [117] Bossard and Ruef showed that all previously known solutions in the almost-BPS class may be recovered from their nilpotent orbit analysis. As for the composite non-BPS black holes in \mathcal{O}_{22} , described above, there are no regular single-centre solutions falling strictly in this orbit. The almost-BPS class does admit regular multi-centre solutions [136, 137], but, unlike the composite non-BPS class, there can be no interactions between space-like non-BPS centres. For an interacting two-centre solution the scalar momentum P at one of the two horizons will lie in the single-centre orbit \mathcal{O}_{18a} of time-like 1/2-BPS black holes given by the Young tableaux in (5.105). Hence, it can be understood as an interacting composite system where one centre is space-like non-BPS while the other is time-like 1/2-BPS.

Again, this can be understood in terms of a representative nilpotent charge $Q \in \mathfrak{p}$ of the principle orbit. Q satisfies $Q^7 = 0$ but not $Q^6 = 0$ and hence there are no regular single-centre solutions. But, as with the composite non-BPS class, it is not difficult to show [117] that Q may be split $Q = Q_1 + Q_2$, where $Q_1 \in \mathcal{O}_{18a}$ and $Q_2 \in \mathcal{O}_{18b}$.

The corresponding class of entangled states given by Kostant-Sekiguchi is, again in common with the subregular \mathcal{O}_{22} orbit, a purely nilpotent family with genuine four-way entanglement:

$$|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle. \quad (5.117)$$

h. \mathcal{O}_{20} : L_{a_4} and composite black holes. This class has six distinct $\mathrm{SO}_0(4,4)$ -orbits, given by the six (labelled) signed Young tableaux,

$$\begin{array}{|c|c|c|c|} \hline - & + & - & + \\ \hline + & & & \\ \hline - & & & \\ \hline + & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline + & - & + & - \\ \hline - & & & \\ \hline + & & & \\ \hline - & & & \\ \hline \end{array}, \quad (I, II \begin{array}{|c|c|c|c|} \hline + & + & + & + \\ \hline + & + & + & + \\ \hline + & + & + & + \\ \hline + & + & + & + \\ \hline \end{array} I, II). \quad (5.118)$$

The system of equations for this class of orbits may be obtained by a straight forward truncation of either the composite non-BPS black holes or the almost-BPS system [117]. However, a complete treatment, clarifying this point, is given in [123]. In fact, this observation is generically true: all systems of equations, and therefore their black hole solutions, can be obtained by suitable truncations from just three of the nilpotent orbits, \mathcal{O}_{18a} , \mathcal{O}_{22} and \mathcal{O}_{24} , corresponding to the time-like 1/2-BPS, composite non-BPS and almost-BPS classes respectively [117].

One of the four-qubit representatives is given by the four-way entangled state,

$$i|0001\rangle + |0110\rangle - i|1011\rangle, \quad (5.119)$$

however, unlike the previous two cases, the family containing this orbit has a semisimple component as can be seen from its representative state,

$$a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle. \quad (5.120)$$

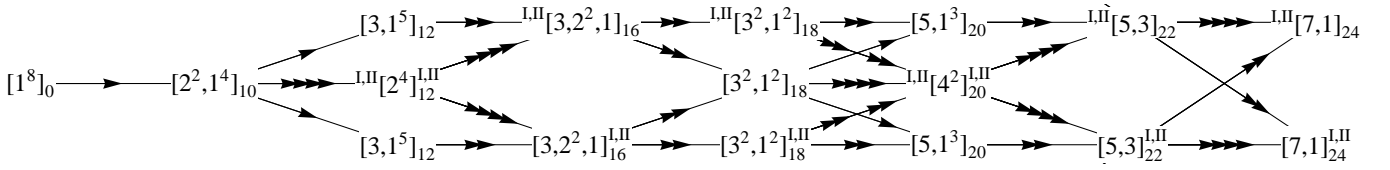


FIG. 8: $SO_0(4, 4)$ Hasse diagram. The integers inside the bracket indicate the structure of the appropriate Young tableau. The subscript indicates the real dimension of the orbit. The arrows indicate their closure ordering defining a partial order [133].

General comments. It is intriguing that the three governing nilpotent orbits are precisely those which constitute entanglement families in their own right, i.e. their semisimple components are identically zero. These same orbits are also interesting from an entanglement perspective. While all the SLOCC invariants are identically zero in these instances, two of the orbits have genuine four-way entanglement. In this sense they are on the same footing as the three-qubit W state. Since the SLOCC invariants (or n^{th} -roots thereof) provide good entanglement measures [62], it is natural to ask what are the physical properties of such entangled states.

In the case of three qubits there is a very precise notion of the “degree” of non-locality (or contextuality) given by [145],

$$A\text{-EPR} < W < \text{GHZ}. \quad (5.121)$$

While both being three-way entangled the GHZ state is *strongly* contextual while the W state is not [145]. This agrees precisely with the entanglement classification obtained using the SLOCC paradigm. Can this hierarchy be generalised to the four-qubit case (this, at first sight at least, seems difficult since the strong contextually condition satisfied by the three-qubit GHZ state is, in a certain sense, maximal) and, if so, will it agree with the SLOCC entanglement classification. Of course, the SLOCC entanglement classification is unambiguously “correct” in its own terms - two states may be probabilistically interrelated by local operations and classical communication if and only if they are in the same SLOCC entanglement class. What we are asking here rather is whether two distinct SLOCC classes of entanglement can always be distinguished by some physical and observable measure of their (generalised) non-locality properties [193]. The simplicity of the two totally entangled orbits, \mathcal{O}_{22} and \mathcal{O}_{24} , coupled with the fact that the strongly contextual GHZ state is in a different orbit altogether, makes them a natural testing ground.

5.6. The attractor mechanism for STU black holes

5.6.1. Attractors

We have seen that there are static spherically symmetric extremal black hole solutions in the STU model of three basic types. There are supersymmetric ($\frac{1}{2}$ BPS) black holes, and nonsupersymmetric (non-BPS) ones with either vanishing or nonvanishing central charge. These can be characterized by either the entanglement properties of the four-qubit state $|\Lambda\rangle$ or by the nilpotent orbits of the associated Noether charge Q . As discussed in the previous subsection the latter procedure results in a finer classification of black holes that can be mapped in a one to one manner to a corresponding *complex* SLOCC classification of nilpotent states of four-qubits.

We have also demonstrated how we can elegantly repackage the information on the charges the moduli, and the warp factor in a *complex* three-qubit state satisfying special reality conditions. The moduli characterize the geometry of the extra dimensions, on the other hand the warp factor characterizes the geometry of the spacetime manifold. From the spacetime perspective the black hole solutions we are interested in are of Reissner-Nordstrom type, with asymptotically Minkowski behavior. As we have noted elsewhere the moduli are really massless scalar fields without any potential. Their dynamics taken together with the warp factor makes it possible to calculate the macroscopic black hole entropy via the Bekenstein-Hawking area law. However, fixing the values of these moduli at the asymptotic region and then solving the dynamical equations governing their radial flow, gives rise to their horizon values. These continuously adjustable asymptotic values would then feature in the macroscopic black hole entropy. This is a dangerous possibility since the entropy should depend only on quantities that take discrete values, such as electric and magnetic charges. This could be a problem for a microscopic reinterpretation of our macroscopic entropy since the number of microstates is an integer that should not depend on continuous parameters.

Luckily the radial dependence of these moduli fields gives rise to *attractors* [24–26]. This means that regardless of their asymptotic values, the moduli flow to particular horizon values that can be expressed in terms of the quantized charges. The existence of such attractors, necessary for a microscopic reinterpretation of the black hole entropy, is the essence of the *attractor mechanism*. In the following we would like to see how the attractor mechanism unfolds itself in terms of our three qubit state of (5.47), as we start from the asymptotic region and approach the horizon.

TABLE III: Each black hole nilpotent $SO_0(4, 4)$ orbit corresponds to a 4-qubit nilpotent $[SL(2, \mathbb{C})]^4$ orbit. z_H is the horizon value of the $\mathcal{N} = 2$, $D = 4$ central charge.

STU black holes			dim $_{\mathbb{R}}$	Four qubits		
description	Young tableaux	$SO_0(4, 4)$ coset		$[SL(2, \mathbb{C})]^4$ coset	nilpotent rep	family
trivial	trivial	$\frac{SO_0(4, 4)}{SO_0(4, 4)}$	1	$\frac{[SL(2, \mathbb{C})]^4}{[SL(2, \mathbb{C})]^4}$	0	$\in G_{abcd}$
doubly-critical $\frac{1}{2}$ BPS		$\frac{SO_0(4, 4)}{[SL(2, \mathbb{R}) \times SO(2, 2, \mathbb{R})] \ltimes [(\mathbf{2}, \mathbf{4})^{(1)} \oplus \mathbf{1}^{(2)}]}$	10	$\frac{[SL(2, \mathbb{C})]^4}{[SO(2, \mathbb{C})]^3 \ltimes \mathbb{C}^4}$	$ 0110\rangle$	$\in L_{abc_2}$
critical, $\frac{1}{2}$ BPS and non-BPS		$\frac{SO_0(4, 4)}{SO(3, 2; \mathbb{R}) \ltimes [(\mathbf{5} \oplus \mathbf{1})^{(2)}]}$	12	$\frac{[SL(2, \mathbb{C})]^4}{[SO(3, \mathbb{C}) \times \mathbb{C}] \times [SO(2, \mathbb{C}) \ltimes \mathbb{C}]}$	$ 0110\rangle + 0011\rangle$	$\in L_{a_2 b_2}$
	$\left(\begin{array}{c} I, II \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \end{array} \\ I, II \end{array} \right)$	$\frac{SO_0(4, 4)}{Sp(4, \mathbb{R}) \ltimes [(\mathbf{5} \oplus \mathbf{1})^{(2)}]}$				
lightlike $\frac{1}{2}$ BPS and non-BPS	$\left(\begin{array}{c} I, II \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \end{array} \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \\ I, II \end{array} \end{array} \right)$	$\frac{SO_0(4, 4)}{SL(2, \mathbb{R}) \ltimes [(2 \times \mathbf{2})^{(1)} \oplus (3 \times \mathbf{1})^{(2)} \oplus \mathbf{2}^{(3)}]}$	16	$\frac{[SL(2, \mathbb{C})]^4}{[SO(2, \mathbb{C}) \ltimes \mathbb{C}] \times \mathbb{C}^2}$	$ 0110\rangle + 0101\rangle + 0011\rangle$	$\in L_{a_2 0_3 \oplus \bar{1}}$
large non-BPS $z_H \neq 0$		$\frac{SO_0(4, 4)}{SO(1, 1, \mathbb{R}) \times SO(1, 1, \mathbb{R}) \ltimes [((\mathbf{2}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}))^{(2)} \oplus \mathbf{1}^{(4)}]}$	18	$\frac{[SL(2, \mathbb{C})]^4}{\mathbb{C}^3}$	$\frac{i}{\sqrt{2}}(0001\rangle + 0010\rangle - 0111\rangle - 1011\rangle)$	$\in L_{ab_3}$
composite		$\frac{SO_0(4, 4)}{SO(2, 1; \mathbb{R}) \ltimes [\mathbf{1}^{(2)} \oplus \mathbf{3}^{(4)} \oplus \mathbf{1}^{(6)}]}$	20	$\frac{[SL(2, \mathbb{C})]^4}{SO(2, \mathbb{C}) \times \mathbb{C}}$	$i 0001\rangle + 0110\rangle - i 1011\rangle$	$\in L_{a_4}$
	$\left(\begin{array}{c} I, II \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \end{array} \\ I, II \end{array} \right)$	$\frac{SO_0(4, 4)}{Sp(2, \mathbb{R}) \ltimes [\mathbf{1}^{(2)} \oplus \mathbf{3}^{(4)} \oplus \mathbf{1}^{(6)}]}$				
large $\frac{1}{2}$ BPS and non-BPS $z_H = 0$	$\left(\begin{array}{c} I, II \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \end{array} \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \\ I, II \end{array} \end{array} \right)$	$\frac{SO_0(4, 4)}{SO(2, \mathbb{R}) \times SO(2, \mathbb{R}) \ltimes [((\mathbf{2}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}))^{(2)} \oplus \mathbf{1}^{(4)}]}$	18	$\frac{[SL(2, \mathbb{C})]^4}{[SO(2, \mathbb{C})]^2 \times \mathbb{C}}$	$ 0000\rangle + 0111\rangle$	$\in L_{0_3 \oplus \bar{1}^0 3 \oplus \bar{1}}$
composite non-BPS	$\left(\begin{array}{c} I, II \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \end{array} \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \\ I, II \end{array} \end{array} \right)$	$\frac{SO_0(4, 4)}{\mathbb{R}^3(2) \oplus \mathbb{R}^1(4) \oplus \mathbb{R}^2(6)}$	22	$\frac{[SL(2, \mathbb{C})]^4}{\mathbb{C}}$	$ 0000\rangle + 0101\rangle + 1000\rangle + 1110\rangle$	$\in L_{0_5 \oplus \bar{3}}$
composite Almost-BPS	$\left(\begin{array}{c} I, II \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \end{array} \\ \begin{array}{c} \text{Young diagram with 4 boxes in a row and 4 boxes in a column} \\ I, II \end{array} \end{array} \right)$	$\frac{SO_0(4, 4)}{\mathbb{R}^{(2)} \oplus \mathbb{R}^{2(6)} \oplus \mathbb{R}^{(10)}}$	24	$\frac{[SL(2, \mathbb{C})]^4}{\text{id}}$	$ 0000\rangle + 1011\rangle + 1101\rangle + 1110\rangle$	$\in L_{0_7 \oplus \bar{1}}$

More precisely our basic concern will be a study of the radial behavior of the three-qubit state

$$|\chi(\tau)\rangle = e^{U(\tau)}(V \otimes V \otimes V)(S_3(\tau) \otimes S_2(\tau) \otimes S_1(\tau))|\gamma\rangle. \quad (5.122)$$

where

$$|\gamma\rangle = (\sigma_3 \otimes \sigma_3 \otimes \sigma_3)|\Gamma\rangle \quad (5.123)$$

obtained from (5.47) after setting $k = p_\sigma = 0$. Here for the definitions see (5.46) and for later use we also give the explicit form of the amplitudes of $|\gamma\rangle$

$$\begin{pmatrix} \gamma_{000}, & \gamma_{001}, & \gamma_{010}, & \gamma_{100} \\ \gamma_{111}, & \gamma_{110}, & \gamma_{101}, & \gamma_{011} \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} p^0, & p^1, & p^2, & p^3 \\ -q_0, & q_1, & q_2, & q_3 \end{pmatrix}. \quad (5.124)$$

We will also use the state $|\psi(\tau)\rangle$ defined as

$$|\psi(\tau)\rangle := e^{-U(\tau)}|\chi(\tau)\rangle \quad (5.125)$$

obtained after removing the dependence on the warp factor. For the amplitudes of the state $|\psi\rangle$ at the horizon we use the shorthand notation $\psi_{ABC} \equiv \lim_{\tau \rightarrow \infty} \psi_{ABC}(\tau)$ with $A, B, C = 0, 1$.

Recall, the effective Lagrangian of the STU model describes the motion of a fiducial particle on $\mathbb{R} \times \mathcal{M}_4$ subject to the potential $-e^{2U}V_{BH}$. Here the four dimensional moduli space is $\mathcal{M}_4 = [\text{SL}(2, \mathbb{R})/\text{SO}(2)]^{\otimes 3}$. The parameter $\tau = 1/r$ plays the role of time. According to (5.53) energy is conserved. Close to local maxima of $e^{2U}V_{BH}$ we expect oscillatory motion, close to minima of $e^{2U}V_{BH}$ we expect unstable solutions *except* when the initial conditions are fine tuned corresponding to extremal black holes. In this case the particle climbs the hill with just enough energy to reach the top ending its motion there. Such finite particle actions correspond to *solitons* which are finite energy solutions of the original field theory. Hence we can regard our extremal black holes as solitons. In order to classify such solutions one has to find the critical points of V_{BH} at the horizon. Using the amplitudes of $\lim_{\tau \rightarrow \infty} |\psi(\tau)\rangle$ extremization of the black-hole potential (5.45) with respect to the moduli yields the following classification of attractor equations [146]:

- **BPS solutions**

$$\psi_1 = \psi_2 = \psi_4 = 0, \quad (5.126)$$

- **Non-BPS solutions with $Z \neq 0$**

$$|\psi_0|^2 = |\psi_1|^2 = |\psi_2|^2 = |\psi_4|^2, \quad (5.127)$$

Notice that the amplitudes $\psi_0 = \psi_{000}$ and $\psi_7 = \psi_{111}$ play a special role as they are related to the central charge and its complex conjugate,

$$Z = -\psi_7, \quad \bar{Z} = \psi_0. \quad (5.128)$$

Since $Z \neq 0$ the corresponding amplitudes are non-zero.

- **Non-BPS solutions with $Z = 0$.**

$$\psi_0 = \psi_1 = \psi_2 = 0 \quad (5.129)$$

and two more cases with 12 replaced by 23 and 31 respectively in the last two amplitudes.

5.6.2. BPS attractors

Let us consider now the case of BPS attractors. In this case since $D_i Z = 2e^{i\alpha}\partial_i|Z|$ according to (5.54) critical points of $|Z|$ are also critical points of V_{BH} . It turns out [24–26] that such critical points are also minima. Recall also from (5.61) that for BPS solutions the second order equations of (5.57) can be replaced by first order ones. If we assume $Z_c \neq 0$ then from the first of (5.61) we get

$$\lim_{\tau \rightarrow \infty} e^{-U} = |Z_c|\tau \quad \text{where} \quad Z_c := \lim_{\tau \rightarrow \infty} Z(\tau). \quad (5.130)$$

Hence from (5.51) the near horizon geometry of the black hole is $AdS_2 \times S^2$

$$ds^2 = \left(-\frac{r^2}{|Z|_c^2} dt^2 + \frac{|Z|_c^2}{r^2} dr^2 \right) + |Z|_c^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad (5.131)$$

Here as usual $\tau = 1/r$ where r is the radial distance from the horizon. The horizon area is

$$A = 4\pi|Z_c|^2 \quad (5.132)$$

hence the thermodynamic Bekenstein-Hawking entropy is

$$S_{BH} = \frac{A}{4} = \pi|Z_c|^2. \quad (5.133)$$

In order to calculate the value of Z_c in terms of the charges let us have a look at the attractor equations (5.126). In the case of the STU model these equations determine the horizon values of the moduli in terms of the charges. These equations also have a very interesting quantum information theoretic interpretation [3]. We see that though the amplitudes of the state $|\psi(\tau)\rangle$ away from the horizon are generally nonzero, at the horizon all of the amplitudes die out except for ψ_{000} and ψ_{111} . According to (2.6)

these are the non-zero amplitudes of the GHZ states: the BPS attractor flow can be interpreted as a distillation procedure for a GHZ state. Moreover, since at the critical point $V_{BH} = |Z|^2$ is minimal, according to (5.49) and (5.125) it follows that at this point the magnitudes of the GHZ amplitudes are minimized. (5.133) also shows that the black hole entropy is proportional to the horizon value of the magnitudes of the GHZ amplitudes of our three-qubit state $|\psi(\tau)\rangle$.

Let us write the superpotential of (5.55) as

$$W(z_3, z_2, z_1) = \gamma_{CBA} c^C b^B a^A = \gamma_{CBA} \varepsilon^{AA'} \varepsilon^{BB'} \varepsilon^{CC'} c_{C'} b_{B'} a_{A'} \quad (5.134)$$

$$a_A \leftrightarrow \begin{pmatrix} 1 \\ z_1 \end{pmatrix}, \quad b_B \leftrightarrow \begin{pmatrix} 1 \\ z_2 \end{pmatrix}, \quad c_C \leftrightarrow \begin{pmatrix} 1 \\ z_3 \end{pmatrix}.$$

Then the (5.126) BPS attractor equations can be written as

$$W(\bar{z}_3, z_2, z_1) = 0, \quad W(z_3, \bar{z}_2, z_1) = 0, \quad W(z_3, z_2, \bar{z}_1) = 0 \quad (5.135)$$

and their complex conjugates. The amplitudes of $|\gamma\rangle$ are given by (5.124).

Using the fact that γ_{CBA} is real these equations taken together with their complex conjugates are equivalent to the vanishing of the 2×2 determinants [3, 70]

$$\text{Det}(\gamma_{CBA} c^C) = 0, \quad \text{Det}(\gamma_{CBA} b^B) = 0, \quad \text{Det}(\gamma_{CBA} a^A) = 0 \quad (5.136)$$

provided the imaginary parts of the moduli are non vanishing. The result is three quadratic equations. Assuming y_1, y_2 and y_3 positive, the stabilized values for the moduli are given by,

$$z^a(\infty; p, q) = \frac{(\gamma_0 \cdot \gamma_1)^a + i\sqrt{-D}}{(\gamma_0 \cdot \gamma_0)^a}, \quad a = 1, 2, 3.$$

$$(\gamma_0 \cdot \gamma_1)^1 \equiv \gamma_{CB0} \varepsilon^{CC'} \varepsilon^{BB'} \gamma_{C'B'1}, \quad (\gamma_0 \cdot \gamma_1)^2 \equiv \gamma_{C0A} \varepsilon^{CC'} \varepsilon^{AA'} \gamma_{C'1A'}, \quad (\gamma_0 \cdot \gamma_1)^3 \equiv \gamma_{0BA} \varepsilon^{BB'} \varepsilon^{AA'} \gamma_{1B'A'} \quad (5.137)$$

with D being Cayley's hyperdeterminant. In order to have such solutions $-D$ should be positive and $(\gamma_0 \cdot \gamma_0)^a$ should be negative. The latter means that the quantities $p^i p^j - p^0 q_k > 0$ where i, j, k are distinct elements of the set $\{1, 2, 3\}$.

Using the stabilized values of the moduli one can calculate the GHZ amplitudes of $|\psi\rangle$. The final form of this three-qubit state is [3]

$$\lim_{\tau \rightarrow \infty} |\psi(\tau)\rangle = (-D)^{1/4} \frac{1}{\sqrt{2}} (e^{-i\alpha} |000\rangle - e^{i\alpha} |111\rangle) \quad (5.138)$$

$$\tan \alpha = \sqrt{-D} \frac{p^0}{2p^1 p^2 p^3 - p^0(p^0 q_0 + p^1 q_1 + p^2 q_2 + p^3 q_3)} \quad (5.139)$$

$$\begin{aligned} D = & (p^0 q_0 + p^1 q_1 + p^2 q_2 + p^3 q_3)^2 \\ & - 4((p^1 q_1)(p^2 q_2) + (p^2 q_2)(p^3 q_3) + (p^3 q_3)(p^1 q_1)) \\ & + 4p^0 q_1 q_2 q_3 - 4q_0 p^1 p^2 p^3 \end{aligned} \quad (5.140)$$

This state is indeed of the unnormalized GHZ form. The norm of this state is proportional to the black hole entropy [1, 3, 70]

$$S_{BH} = \pi \sqrt{-D}. \quad (5.141)$$

Notice that apart from reproducing the result of (2.9) we also managed to understand the behavior of BPS attractors in terms of a *complex* three-qubit state depending on the charges *and the moduli*. It is also interesting that the phase of the GHZ amplitudes, i.e. the phase of the central charge at the horizon, can be expressed in terms of the Freudenthal dual [41] amplitude \hat{p}^0 as [3]

$$\tan \alpha = \frac{p^0}{\hat{p}^0}. \quad (5.142)$$

We remark in closing that for non-BPS solutions with vanishing central charge the attractor states will be again of the (5.138) form with the important difference that now the basis vectors of the states are changed [30] by flipping one of the bits via the use of the bit flip operator σ_1 . In this case modifying (5.138) one can now obtain GHZ states by combining the basis vectors $|100\rangle$ and $|011\rangle$ etc.

5.6.3. The distillation procedure for non-BPS attractors

In this section we would like to demonstrate how the radial flow for the moduli gives rise to the distillation of a special three-qubit (attractor) state at the black hole horizon [30]. Here we demonstrate this process for the non-BPS seed solutions [115] of the class of (5.127). In this case of the eight charges only the four $q_0 < 0$ and $p^i > 0$, $i = 1, 2, 3$ are switched on. If we are embedding the STU model into type IIA string theory the interpretation of these charges is given in terms of wrapping numbers of $D0$ and $D4$ branes on the 0 and 4 cycles of the six dimensional torus T^6 . In this case the basic U-duality invariant is again related to Cayley's hyperdeterminant calculated for the four charge version of the state of (5.124). Its value is $D = 4q_0 p^1 p^2 p^3$. Using definition (5.17) the equations to be solved are given by (5.71)-(5.73). The solution of these equations can be expressed in terms of harmonic functions \hat{h}_I , $I = 0, 1, 2, 3$ as [115]

$$e^{\alpha_i + \alpha_j - \alpha_k} = \frac{1}{d_k + D^{1/4}\tau} \equiv \frac{1}{\hat{h}_k}, \quad i \neq j \neq k, \quad i, j, k = 1, 2, 3 \quad (5.143)$$

$$e^{-\beta} = -\hat{h}_0 - \frac{B^2}{\hat{h}_1 \hat{h}_2 \hat{h}_3}, \quad h_0 \equiv -d_0 - D^{1/4}\tau \quad (5.144)$$

where

$$d_i = \frac{D^{1/4}}{\sqrt{2}p^i}, \quad d_0 = -\frac{D^{1/4}}{\sqrt{2}q_0}(1 + B^2) \quad (5.145)$$

For this 5 parameter solution we have

$$z^i = R_i \frac{B - ie^{-2U}}{\frac{1}{2}|\epsilon_{ijk}|\hat{h}_j\hat{h}_k}, \quad e^{-4U} = -\hat{h}_0\hat{h}_1\hat{h}_2\hat{h}_3 - B^2, \quad R_i = \sqrt{\frac{-2q_0 p^i}{|\epsilon_{ijk}|p^j p^k}}. \quad (5.146)$$

Notice that $B \equiv x^1(0) = x^2(0) = x^3(0)$ describes the real part for the asymptotic values of the moduli.

Given the radial dependence of the moduli and the warp factor one can calculate the Fourier amplitudes of the state $|\tilde{\chi}\rangle = (H \otimes H \otimes H)|\chi\rangle$. This state coming from (5.47) can now display how the attractor mechanism as a distillation procedure unfolds. After removing the warp factor as in (5.125) one can examine the behavior of the three-qubit state $|\tilde{\psi}(\tau)\rangle = (H \otimes H \otimes H)|\psi(\tau)\rangle$ in the asymptotic region and on the horizon. In the asymptotic Minkowski region we get

$$\begin{aligned} \lim_{\tau \rightarrow 0} |\tilde{\psi}(\tau)\rangle = & \frac{1}{\sqrt{2}}(p^1|001\rangle + p^2|010\rangle + p^3|100\rangle - iB(p^2 + p^3)|110\rangle \\ & - iB(p^1 + p^3)|101\rangle - iB(p^1 + p^2)|011\rangle \\ & + [q_0 - B^2(p^1 + p^2 + p^3)]|111\rangle). \end{aligned} \quad (5.147)$$

On the other hand at the horizon one obtains

$$\lim_{\tau \rightarrow \infty} |\tilde{\psi}(\tau)\rangle = (-4q_0 p^1 p^2 p^3)^{1/4} \frac{1}{2} (|001\rangle + |010\rangle + |100\rangle - |111\rangle). \quad (5.148)$$

This result shows that if we “start” asymptotically with the state of (5.147) with seven nonvanishing generically different amplitudes we end up with the state of (5.148) having merely four nonvanishing amplitudes that are the same up to a sign. Notice that although the asymptotic state features B , the asymptotic value for the moduli, the horizon state is independent of this quantity. Hence, in accordance with the attractor mechanism, different values for B defining different initial three-qubit states flow to the *same* three-qubit attractor state determined only by the charges.

There is an important subtlety here. For BPS black holes no asymptotic values for the moduli can appear in their horizon values. Indeed, according to (5.138)-(5.140) the states on the horizon can be expressed entirely in terms of the charges. Based on our results as represented by (5.147)-(5.148) for the seed solution one would conclude that a similar result holds also for non-BPS solutions. However, there are special directions given by particular asymptotic conditions which give rise to radial flows resulting in special values of asymptotic moduli at the black hole event horizon. These directions are referred to as *flat* [28, 115, 147, 148]. In other words, the attractor mechanism fixes all the moduli for a BPS charge vector, but may leave flat directions for a non-BPS charge configuration.

An analysis [30] based on the most general non-BPS solution [28] with nonvanishing central charge, as obtained from the seed solution, shows that, for the $D0 - D4$ configuration with the most general asymptotic conditions, the radial flow gives rise

to the following three-qubit state on the horizon,

$$\begin{aligned} \lim_{\tau \rightarrow \infty} |\tilde{\psi}(\tau)\rangle = & \frac{1}{2} \frac{(-4q_0 p^1 p^2 p^3)^{\frac{1}{4}}}{\sqrt{\cosh(\alpha_3) \cosh(\alpha_2) \cosh(\alpha_1)}} \times \\ & \left[-|111\rangle + \cosh(\alpha_1)|001\rangle + \cosh(\alpha_2)|010\rangle + \cosh(\alpha_3)|100\rangle \right. \\ & \left. + i \sinh(\alpha_1)|110\rangle + i \sinh(\alpha_2)|101\rangle + i \sinh(\alpha_3)|011\rangle \right]. \end{aligned} \quad (5.149)$$

Here the real constants α_i , $i = 1, 2, 3$ satisfy the constraint

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \quad (5.150)$$

and, showing up in the asymptotic form of the moduli, account for the flat directions. In the type IIA picture the α_i parameterize deformations of the six torus that preserve its overall volume. We see that for $\alpha_i = 0$ we get back to the result of (5.148), but for the most general case of the non-BPS attractor flow these asymptotic moduli make an appearance in the attractor state.

One can consider the dual case of the D2-D6 charge-configuration [28, 149, 150]. In this case $p^0 > 0$ and $q_a > 0$. The corresponding attractor state is [30]

$$\begin{aligned} \lim_{\tau \rightarrow \infty} |\tilde{\psi}(\tau)\rangle = & \frac{1}{2} \frac{(4p^0 q_1 q_2 q_3)^{\frac{1}{4}}}{\sqrt{\cosh(\alpha_3) \cosh(\alpha_2) \cosh(\alpha_1)}} \times \\ & \left[\sinh(\alpha_1)|001\rangle + \sinh(\alpha_2)|010\rangle + \sinh(\alpha_3)|100\rangle \right. \\ & \left. - i|000\rangle + i \cosh(\alpha_1)|110\rangle + i \cosh(\alpha_2)|101\rangle + i \cosh(\alpha_3)|011\rangle \right]. \end{aligned} \quad (5.151)$$

Notice that for vanishing flat directions the $D0 - D4$ and $D2 - D6$ cases are dual in the sense that their attractor states are related by the bit flip operation $\sigma_1 \otimes \sigma_1 \otimes \sigma_1$.

It is interesting to analyse the effect of the asymptotic data on the attractor states at the horizon. In the $D2 - D6$ case let us change the *signs* of the charges q_1, q_2, q_3 in such a way that the combination $p^0 q_1 q_2 q_3$ showing up in the entropy formula is left invariant. Then one can show [29] that the attractor state is

$$|\tilde{\psi}\rangle_{m_1 m_2 m_3} = i(4p^0 q_1 q_2 q_3)^{1/4} \frac{1}{2} [m_1|110\rangle + m_2|101\rangle + m_3|011\rangle - |000\rangle] \quad (5.152)$$

where

$$(m_1, m_2, m_3) \in \{(+, +, +), (+, -, -), (-, +, -), (-, -, +)\}. \quad (5.153)$$

Hence, although the changes in sign do not change the black hole entropy, they have an effect on the particular form of the state. The possible sign changes can be implemented via the action of the phase flip error operators $\sigma_3 \otimes \sigma_3 \otimes I$ plus cyclic permutations. Notice that the Fourier transformed state $|\psi\rangle$ features the bit flip operators σ_1 in a similar combination.

Let us now fix the signs of the charges of the $D2 - D6$ system, and vary the values for the asymptotic parameters α_i responsible for the flat directions. Let us also refer to the state of (5.151) as $|\tilde{\psi}\rangle_{\alpha_3 \alpha_2 \alpha_1}$. One can then see that by virtue of (5.150)

$$|\tilde{\psi}\rangle_{\alpha_3 \alpha_2 \alpha_1} = (E_3 \otimes E_2 \otimes E_1) |\tilde{\psi}\rangle_{+++} \quad (5.154)$$

where

$$E_i = \frac{1}{\sqrt{\cosh \alpha_i}} \begin{pmatrix} 1 & 0 \\ i \sinh \alpha_i & \cosh \alpha_i \end{pmatrix}. \quad (5.155)$$

Hence the changes on the state originating from the flat directions have the interpretation of errors of more general kind depending on continuously changing parameters. It is amusing to see that though the error operators E_i act locally, due to the constraint of (5.150), they are not independent. In quantum information such constraints usually refer to an agreement between the parties affected via the use of classical channels [4, 14].

These results clearly show the relevance of ideas known from the theory of quantum error correcting codes. Taken together with our earlier observations that the structure of the continuous U-duality group $E_{7(7)}$ can be elegantly described via the use of the Hamming code shows that these mathematical coincidences are worth exploring further. Indeed, the tripartite entanglement of seven qubits interpretation shows that for the most general 56 charge black hole configurations there are U-duality transformations of two kind. Either they transform within any of the seven possible STU sectors, or transform the sectors among

themselves. U-duality transformations of the latter form include elements belonging to the Weyl group $W(E_7)$. For example, the order seven automorphism (3.14) that rotates the STU sectors into each other. As we have seen such transformations can be represented in the form featuring CNOT gates. On the other hand U-duality transformations of the former incorporate naturally bit and phase flip error operations operating in the solution space of a fixed STU truncation. These considerations might be indications that the black hole entropy formulae in the semiclassical limit represent some sort of effective “area codes” protecting the information encoded in wrapping configurations of branes from errors arising from fluctuations of the asymptotic data of moduli. We explore this idea in some detail in the next section.

6. ENTANGLEMENT FROM EXTRA DIMENSIONS

In the previous sections we formally employed unnormalized qubits and qutrits to understand the structure of the semiclassical Bekenstein-Hawking entropy and the phenomenon of moduli stabilization in entanglement terms. In all of our considerations we have used an effective four or five dimensional picture given by the relevant classical supergravities. Hence, thus far we have not really made use of the higher dimensional picture that string theory provides. Indeed superstring (M-theory) lives in ten (eleven) spacetime dimensions so it would be desirable to embed our findings in an extra dimensional setting. Moreover, one might hope that via this embedding many aspects of our considerations will be illuminated by the new perspective offered by inclusion of the extra dimensions.

Apart from these interesting possibilities, there is a more down-to-earth reason to embark upon exploring these ideas. Namely, looking at, for example, (5.148) it is not at all obvious what symbols like $|001\rangle$ actually mean in the string context. Of course these symbols refer to the basis vectors of some sort of Hilbert space. But how this Hilbert space is defined? Is this Hilbert space independent of the moduli, or should we rather consider a *family* of such spaces parametrized by them?

In this section we address these important issues. First in the next section we provide a physical basis for connecting the qubits and qutrits to the structure of the extra dimensions. The main idea [50] is to relate them to wrapping configurations of membranes around the extra dimensions. In later sections we give a precise meaning to such issues, by identifying the underlying Hilbert space giving home to the qubits within the cohomology of the extra dimensions [51]. We will see that as an extra bonus these considerations also give a rationale for our use of charge and moduli dependent three-qubit states in the previous sections. In the following for simplicity we consider toroidal compactifications and use the type IIB duality frame, meaning that we formulate our ideas within the framework of the corresponding string theory.

6.1. Wrapped branes as qubits

The microscopic string-theoretic interpretation of the charges is given by configurations of intersecting D3-branes, wrapping around the six compact dimensions T^6 . The 3-qubit basis vectors $|ABC\rangle$ are associated with the corresponding eight wrapping cycles. In particular, one can relate a well-known fact of quantum information theory, that the most general real three-qubit state up to local unitaries can be parameterised by four real numbers and an angle, to a well-known fact of string theory, that the most general STU black hole can be described by four D3-branes intersecting at an angle.

The microscopic analysis is not unique since there are many ways of embedding the STU model in string/M-theory, but a useful example from our point of view is that of four D3-branes of Type IIB wrapping the (579), (568), (478), (469) cycles of T^6 with wrapping numbers N_0, N_1, N_2, N_3 and intersecting over a string [151]. The wrapped circles are denoted by crosses and the unwrapped circles by noughts as shown in Table IV. This picture is consistent with the interpretation of the 4-charge black hole as bound state at threshold of four 1-charge black holes [27, 152, 153]. The fifth parameter θ is obtained [97, 154] by allowing the N_3 brane to intersect at an angle which induces additional effective charges on the (579), (569), (479) cycles.

To make the black hole/qubit correspondence we associate the three T^2 with the $SL(2)_A \times SL(2)_B \times SL(2)_C$ of the three qubits Alice, Bob, and Charlie. The eight different cycles then yield eight different basis vectors $|ABC\rangle$ as in the last column of Table IV, where $|0\rangle$ corresponds to xO and $|1\rangle$ to OX. To wrap or not to wrap; that is the qubit. We see immediately that we reproduce the five parameter three-qubit state $|\Psi\rangle$ of (7.9):

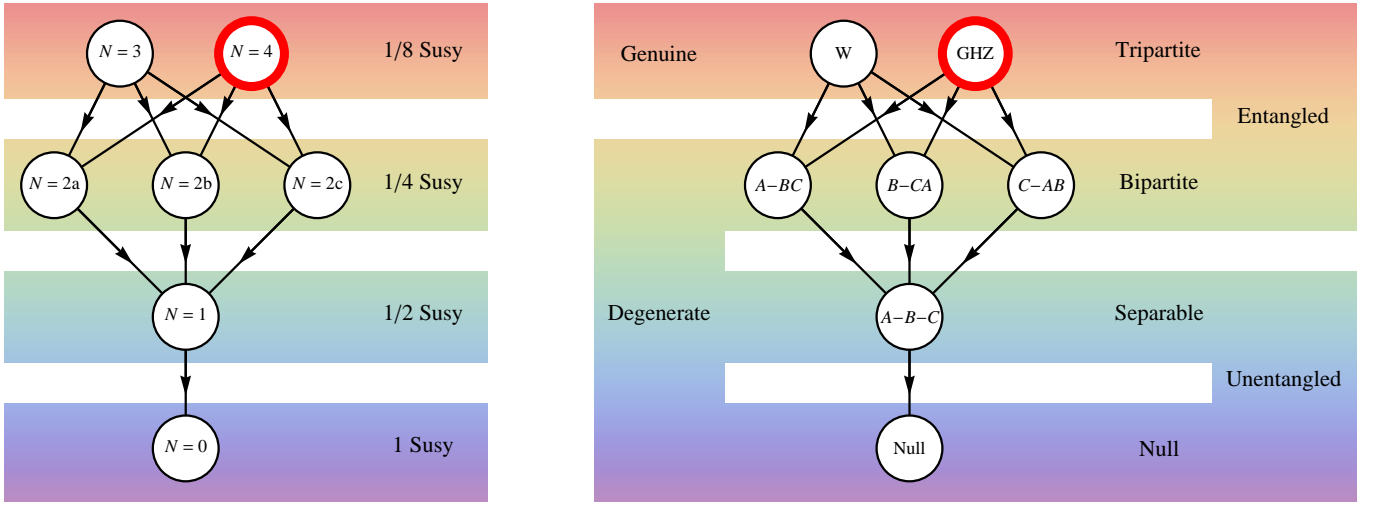
$$\begin{aligned} |\Psi\rangle = & -N_3 \cos^2 \theta |001\rangle - N_2 |010\rangle + N_3 \sin \theta \cos \theta |011\rangle \\ & - N_1 |100\rangle - N_3 \sin \theta \cos \theta |101\rangle + (N_0 + N_3 \sin^2 \theta) |111\rangle. \end{aligned} \quad (6.1)$$

Note the GHZ state describes four D3-branes intersecting over a string. By embedding this picture in $\mathcal{N} = 8$ supergravity we find an exact correspondence between the possible intersections preserving different degree of supersymmetry and the entanglement classes of 3-qubits as illustrated in Figure 10.

4	5	6	7	8	9	macro charges	micro charges	$ ABC\rangle$
x	o	x	o	x	o	p^0	0	$ 000\rangle$
o	x	o	x	x	o	q_1	0	$ 110\rangle$
o	x	x	o	o	x	q_2	$-N_3 \sin \theta \cos \theta$	$ 101\rangle$
x	o	o	x	o	x	q_3	$N_3 \sin \theta \cos \theta$	$ 011\rangle$
o	x	o	x	o	x	q_0	$N_0 + N_3 \sin^2 \theta$	$ 111\rangle$
x	o	x	o	o	x	$-p^1$	$-N_3 \cos^2 \theta$	$ 001\rangle$
x	o	o	x	x	o	$-p^2$	$-N_2$	$ 010\rangle$
o	x	x	o	x	o	$-p^3$	$-N_1$	$ 100\rangle$

TABLE IV: Three qubit interpretation of the 8-charge $D = 4$ black hole from four D3-branes wrapping around the lower four cycles of T^6 with wrapping numbers N_0, N_1, N_2, N_3 and then allowing N_3 to intersect at an angle θ .

FIG. 9: Left: Classification of $\mathcal{N} = 8$ black holes according to susy. N denotes the number of intersecting D-branes in the microscopic picture. Right: The entanglement classification of 3-qubits. The arrows represent the removal of a D-brane or a non-invertible SLOCC operation.



6.2. Qubits from extra dimensions

In the previous section we have seen that wrapped branes can be used to realize qubits, the basic building blocks used in quantum information. Based on these findings it is natural to expect that such brane configurations wrapped on different cycles of the manifold of extra dimensions should be capable of accounting for many more of the surprising findings of the BHQC. The aim of the present section is to show that by simply reinterpreting some of the well-known results of toroidal compactification of type IIB string theory in a quantum information theoretic fashion this expectation can indeed be justified. In particular we identify the Hilbert space giving home to the qubits inside the cohomology of the extra dimensions, establishing for the phrase “to wrap or not to wrap, that is the qubit” a mathematical meaning. In order to do this let us recall some basic ideas of Calabi-Yau compactifications of type II string theories [78].

Calabi-Yau compactifications play an important role in string theory. Calabi-Yau spaces are six dimensional complex Kähler manifolds that are Ricci flat. They are promising candidates for producing sensible four dimensional low energy phenomenology. Moreover due to their special structure they are ideal playing grounds for obtaining results illustrating the interplay between the geometry of extra dimensions and ordinary four-dimensional spacetime as tested by extended objects. As we have seen in sections discussing the attractor mechanism there are scalar fields called moduli describing the fluctuations of such spaces. For type II string theories such moduli can be separated into two characteristic types. Either they are associated with the deformations of the complex structure or with the Kähler structure of the underlying Calabi-Yau. For type IIA supergravity the moduli are associated with the latter and for type IIB with the former. In the following we merely consider the special case of toroidal compactifications, hence we assume that the geometry of the extra dimensions is that of a six-torus T^6 . In the type IIA picture Kähler structure deformations correspond to deformations of the size, on the other hand in the type IIB picture complex structure deformations alter the shape. We adopt the type IIB picture, hence by moduli in the following we mean the complex structure. Restricting further to the special case of regarding T^6 as $T^2 \times T^2 \times T^2$ as in the previous subsection, we can then associate three-qubits with the three two dimensional tori.

The moduli are now $z^j = x^j - iy^j$ with $y^j > 0$ as in (5.2). They correspond to the deformation parameters for the tori as labelled by $j = 1, 2, 3$. Let us denote the six real coordinates of T^6 by $0 \leq u^j \leq 1$ and $0 \leq v^j \leq 1$. We can imagine these coordinates as the ones adapted to the deformed tori. This means that regarding T^6 as a complex manifold the complex coordinates w^j of T^6 should depend on the moduli z^j . Notice that $T^6 = T^2 \times T^2 \times T^2$ is a very special case since now the number of complex moduli equals the number of complex dimensions of the manifold. For a general Calabi-Yau this need not be the case, since the number of complex moduli [78] is then given by the Hodge number $h^{2,1}$. Next we write

$$w^j = u^j + z^j v^j. \quad (6.2)$$

Hence the real pairs (u^j, v^j) and (x^j, y^j) are coordinates for T^6 and its moduli space $\mathcal{M} = [\text{SL}(2, \mathbb{R}) / \text{SO}(2)]^{\otimes 3}$ respectively. Alternatively w^j and z^j are the corresponding complex coordinates for these spaces.

For a Calabi-Yau space we have a nowhere vanishing holomorphic three-form. For our torus it is

$$\Omega_0 = dw^1 \wedge dw^2 \wedge dw^3. \quad (6.3)$$

We have

$$\int_{T^6} \Omega_0 \wedge \bar{\Omega}_0 = i(8y^1 y^2 y^3) = ie^{-K}, \quad (6.4)$$

where K is the Kähler potential of (5.4) giving rise to the metric $G_{i\bar{j}}$ of (5.3) on the special Kähler manifold \mathcal{M} . Let us now define the nonholomorphic three-form Ω as

$$\Omega \equiv e^{K/2} \Omega_0. \quad (6.5)$$

Define flat covariant derivatives D_i acting on Ω as

$$D_i \Omega = (\bar{z}^i - z^i) D_i \Omega = (\bar{z}^i - z^i) \left(\partial_i + \frac{1}{2} \partial_i K \right) \Omega, \quad (6.6)$$

where $\partial_i = \partial / \partial z^i$. Now one has

$$\Omega = e^{K/2} dw^1 \wedge dw^2 \wedge dw^3, \quad \bar{\Omega} = e^{K/2} d\bar{w}^1 \wedge d\bar{w}^2 \wedge d\bar{w}^3 \quad (6.7)$$

$$D_{\bar{1}} \Omega = e^{K/2} d\bar{w}^1 \wedge dw^2 \wedge dw^3, \quad \bar{D}_{\bar{1}} \bar{\Omega} = e^{K/2} dw^1 \wedge d\bar{w}^2 \wedge d\bar{w}^3 \quad (6.8)$$

with the remaining covariant derivatives obtained via cyclic permutation. Let us consider the action of the Hodge star on our basis of three-forms as given by (6.7)-(6.8). For a form of (p, q) type the action of the Hodge star is defined as

$$(\varphi, \varphi) \frac{\omega^n}{n!} = \varphi \wedge \star \bar{\varphi} \quad (6.9)$$

where for our T^6 in accord with our conventions

$$\omega = i(d\bar{w}^1 \wedge dw^1 + d\bar{w}^2 \wedge dw^2 + d\bar{w}^3 \wedge dw^3). \quad (6.10)$$

Hence we get

$$\star \Omega = i\Omega, \quad \star \bar{\Omega} = -i\bar{\Omega} \quad (6.11)$$

$$\star D_{\bar{a}} \Omega = -i D_{\bar{a}} \Omega, \quad \star \bar{D}_{\bar{a}} \bar{\Omega} = i \bar{D}_{\bar{a}} \bar{\Omega} \quad (6.12)$$

Now we regard the 8 complex dimensional untwisted primitive part [155] of the 20 dimensional space $H^3(T^6, \mathbb{C}) \equiv H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$ equipped with the Hermitian inner product

$$\langle \varphi | \eta \rangle \equiv \int_{T^6} \varphi \wedge \star \bar{\eta} \quad (6.13)$$

as a Hilbert space isomorphic to $\mathcal{H} \equiv (\mathbb{C}^2)^{\otimes 3} \simeq \mathbb{C}^8$ of three qubits. In order to set up the correspondence between the three-forms and the basis vectors of the three-qubit system we use the *negative* of the basis vectors $\Omega, D_{\bar{1}} \Omega, \dots$ multiplied by the imaginary unit i . Changing the order of the one-forms we have, for example, $-i D_{\bar{1}} \Omega = ie^{K/2} dz^3 \wedge dz^2 \wedge d\bar{z}^1$. Hence, we can

take as its representative basis qubit state $|001\rangle$, which corresponds to the usual binary labelling provided we label the qubits from the right to the left.

Using these conventions the basis states of our computational base are given by,

$$-i\Omega \leftrightarrow |000\rangle, \quad -iD_1\Omega \leftrightarrow |001\rangle, \quad -iD_2\Omega \leftrightarrow |010\rangle, \quad -iD_3\Omega \leftrightarrow |100\rangle \quad (6.14)$$

$$-i\bar{\Omega} \leftrightarrow |111\rangle, \quad -i\bar{D}_1\Omega \leftrightarrow |110\rangle, \quad -i\bar{D}_2\Omega \leftrightarrow |101\rangle, \quad -i\bar{D}_3\Omega \leftrightarrow |011\rangle. \quad (6.15)$$

Now the locations of the “1”s correspond to the slots where complex conjugation is effected. One can check that the states above form a basis with respect to the inner product of (6.13) with the usual set of properties in terms of the three-qubit interpretation.

A further check shows that the action of the flat covariant derivatives $D_{\bar{a}}, j = 1, 2, 3$ corresponds to the action of the projective bit flips of the form $I \otimes I \otimes \sigma_+$, $I \otimes \sigma_+ \otimes I$ and $\sigma_+ \otimes I \otimes I$, where I is the 2×2 identity matrix. For the conjugate flat covariant derivatives σ_+ has to be replaced by σ_- . Moreover, the diagonal action of the Hodge star in the computational base is represented by the corresponding action of the *negative* of the parity check operator $i\sigma_3 \otimes \sigma_3 \otimes \sigma_3$.

Now for a three-form representing the cohomology class of a wrapped $D3$ brane configuration we take

$$\psi = p^I \alpha_I - q_I \beta^I \in H^3(T^6, \mathbb{Z}) \quad (6.16)$$

and

$$\alpha_0 = du^1 \wedge du^2 \wedge du^3, \quad \beta^0 = -dv^1 \wedge dv^2 \wedge dv^3 \quad \alpha_1 = dv^1 \wedge du^2 \wedge du^3, \quad \beta^1 = du^1 \wedge dv^2 \wedge dv^3 \quad (6.17)$$

with the remaining four obtained via cyclic permutation. With the canonical choice of orientation we have $\int_{T^6} \alpha_I \wedge \beta^J = \delta_I^J$.

It is well-known [143] that in the Hodge diagonal basis we can express this as

$$\psi = iZ(\psi)\bar{\Omega} - ig^{j\bar{k}}D_j Z(\psi)\bar{D}_{\bar{k}}\bar{\Omega} + \text{c.c.} = iZ(\psi)\bar{\Omega} - i\delta^{\hat{j}\hat{k}}D_{\hat{j}}Z(\psi)\bar{D}_{\hat{k}}\bar{\Omega} + \text{c.c.} \quad (6.18)$$

Here $Z(\psi) = \int_{T^6} \psi \wedge \Omega$ is the central charge. For its explicit expression see (5.55).

Employing our basic correspondence between three-forms and three-qubit states of (6.14)-(6.15) we can write $\psi \leftrightarrow |\psi\rangle$ where

$$|\psi\rangle = \psi_{000}|000\rangle + \psi_{001}|001\rangle + \cdots + \psi_{110}|110\rangle + \psi_{111}|111\rangle \quad (6.19)$$

$$\psi_{111} = -e^{K/2}W(z^3, z^2, z^1) = -\bar{\psi}_{000}, \quad \psi_{001} = -e^{K/2}W(\bar{z}^3, \bar{z}^2, z^1) = -\bar{\psi}_{110} \quad (6.20)$$

and the remaining amplitudes are given by cyclic permutation. Now one can check that, using the definitions in (5.122) and (5.46), the state $|\psi\rangle$ can be written in the form given by (5.125).

Hence our state $|\psi\rangle$ seems to be exactly the same as the one that has already appeared in our previous considerations. It is important to realize however, the basic difference in interpretation. Until now the state $|\psi\rangle$ was merely a *charge* and *moduli* dependent state connected to the 4 dimensional setting of the STU model. Moreover, in that setting the basis states $|ABC\rangle$ with $A, B, C = 0, 1$ had no obvious physical meaning. They merely served as basis vectors providing a suitable frame for a three-qubit reformulation.

Now $|\psi\rangle$ depends on the *charges*, the *moduli* and the *coordinates of the extra dimensions*. Hence, this state is connected to a 10-dimensional setting of the STU model in the type IIB duality frame. Now the basis vectors $|ABC\rangle$ have an obvious physical meaning: they are the Hodge diagonal complex basis vectors of the untwisted primitive part of the third cohomology group of the extra dimensions, i.e. of $H^3(T^6, \mathbb{C})$. They are also basis vectors of a genuine Hilbert space equipped with a natural Hermitian inner product (6.13), isomorphic to the usual space of three qubits. The basis vectors $|ABC\rangle$ not only depend on the coordinates of the tori, but also on the *moduli*. Hence the notation should reflect that $|ABC\rangle$ refers to a *parametrized family* of basis vectors. Since the notation $|ABC; z^1, z^2, z^3\rangle$ is rather awkward we omit the z^i and tacitly assume an implicit dependence on them.

In the new formalism the state $|\psi\rangle$ also has a nice physical meaning. It is the Poincaré dual of the homology cycle representing wrapped $D3$ -brane configurations. Moreover $|\psi\rangle$ can be represented in two different forms. Namely as in (6.19) which is an expansion in a Hodge-diagonal moduli dependent complex base, or in an equivalent way based on the qubit version of (6.16) which is a Hodge-non-diagonal but moduli independent real base.

These results also shed some light on the phenomenon that the fluctuations of the moduli are related to phase flip and bit flip errors we encountered at the end of Section 5.6.3. Indeed, now we see that on the basis vectors $|ABC\rangle$ the flat covariant derivatives with respect to the moduli act naturally as elementary error operations. As a byproduct of this the attractor equations can be given an alternative explanation. For the attractor flow we have a three-qubit state with amplitudes associated with all of the *moduli dependent* basis vectors $|ABC\rangle$. We have seen that at the horizon only special combinations of the amplitudes

associated with special basis vectors survive. For example according to (5.138) for the BPS flow we have only amplitudes multiplied by the basis states $|000\rangle$ and $|111\rangle$. This means that at the horizon bit flip errors of the form $\sigma_1 \otimes I \otimes I$ and their cyclic permutations acting on these basis vectors are suppressed [29]. These conditions can be expressed precisely in the form of the BPS attractor equations i.e. (5.126) and their conjugates. This result is a quantum information theoretic reinterpretation of the well-known property of supersymmetric attractors in the type IIB picture namely that in this case only the $H^{3,0}$ and $H^{0,3}$ parts of the cohomology survive [78].

6.3. Fermionic entanglement from extra dimensions

As a generalization of our considerations giving rise to qubits we consider the problem of obtaining entangled systems of a more general kind from toroidal compactification [51]. The trick is to embed our simple systems featuring few qubits into larger ones. Here we discuss the natural generalization of embedding qubits (based on entangled systems with distinguishable constituents) into fermionic systems (based on entangled systems with indistinguishable ones [156, 157]). In the quantum information theoretic context this possibility has already been elaborated [20], see section 7.7.5. Here we show that toroidal compactifications also incorporate this idea quite naturally.

As in the special case of the STU model we choose analytic coordinates for the complex torus such that the holomorphic one-forms are defined as $dw^i = du^i + z^{ij} dv^j$ where now z^{ij} , $1 \leq i, j \leq 3$ is the period matrix of the torus with the convention

$$z^{ij} = x^{ij} - iy^{ij}. \quad (6.21)$$

For principally polarized Abelian varieties we have the additional constraints

$$z^{ij} = z^{ji}, \quad y^{ij} > 0. \quad (6.22)$$

We choose as usual $\Omega_0 = dw^1 \wedge dw^2 \wedge dw^3$, and the canonical orientation.

Now we exploit the full 20 dimensional space of $H^3(T^6, \mathbb{C})$. We expand $\psi \in H^3(T^6, \mathbb{C})$ in the basis

$$\alpha_0 = du^1 \wedge du^2 \wedge du^3, \quad \alpha_{ij} = \frac{1}{2} \varepsilon_{ii'j'} du^{i'} \wedge du^{j'} \wedge dv^j \quad (6.23)$$

$$\beta^0 = -dv^1 \wedge dv^2 \wedge dv^3, \quad \beta^{ij} = \frac{1}{2} \varepsilon_{jj'i'} du^i \wedge dv^{i'} \wedge dv^{j'}. \quad (6.24)$$

One can then show that

$$\Omega_0 = \alpha_0 + z^{ij} \alpha_{ij} + z^{\#}_{ij} \beta^{ij} - (\text{Det} z) \beta^0 \quad (6.25)$$

where $z^{\#}$ is the transposed cofactor matrix satisfying $zz^{\#} = \text{Det}(z) \mathbf{1}$. One can check that the generalization of the identity of (6.4) holds and $e^{-K} = 8 \text{Det} y$.

An element ψ of $H^3(T^6, \mathbb{Z})$ can be expanded as

$$\psi = p^0 \alpha_0 + P^{ij} \alpha_{ij} - Q_{ij} \beta^{ij} - q_0 \beta^0. \quad (6.26)$$

We can rewrite this as

$$\psi = \frac{1}{3!} \gamma_{ABC} f^A \wedge f^B \wedge f^C \quad (6.27)$$

where

$$(f^1, f^2, f^3, f^4, f^5, f^6) \equiv (f^1, f^2, f^3, f^{\bar{1}}, f^{\bar{2}}, f^{\bar{3}}) = (du^1, du^2, du^3, dv^1, dv^2, dv^3). \quad (6.28)$$

Here ψ_{ABC} has been generalized to a completely antisymmetric tensor of rank three with 20 independent components. Clearly the independent components of γ_{ABC} are identified with the 20 quantized charges $(p^0, P^{ij}, Q_{ij}, q_0)$ related to wrapping three-branes on the corresponding homology cycles. The explicit identification is given by

$$p^0 = \gamma_{123}, \quad \begin{pmatrix} P^{11} & P^{12} & P^{13} \\ P^{21} & P^{22} & P^{23} \\ P^{31} & P^{32} & P^{33} \end{pmatrix} = \begin{pmatrix} \gamma_{23\bar{1}} & \gamma_{23\bar{2}} & \gamma_{23\bar{3}} \\ \gamma_{31\bar{1}} & \gamma_{31\bar{2}} & \gamma_{31\bar{3}} \\ \gamma_{12\bar{1}} & \gamma_{12\bar{2}} & \gamma_{12\bar{3}} \end{pmatrix} \quad (6.29)$$

$$q^0 = \gamma_{1\bar{2}\bar{3}}, \quad \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} = - \begin{pmatrix} \gamma_{1\bar{2}\bar{3}} & \gamma_{1\bar{3}\bar{1}} & \gamma_{1\bar{1}\bar{2}} \\ \gamma_{2\bar{2}\bar{3}} & \gamma_{2\bar{3}\bar{1}} & \gamma_{2\bar{1}\bar{2}} \\ \gamma_{3\bar{2}\bar{3}} & \gamma_{3\bar{3}\bar{1}} & \gamma_{3\bar{1}\bar{2}} \end{pmatrix}. \quad (6.30)$$

Using the language of fermionic entanglement [156, 157] ψ can also be regarded as an unnormalized three fermion state with six single particle states [20], as described in section 7.7.5.

Now we introduce the new moduli dependent basis vectors

$$e^A = f^{A'} S_{A'}^A, \quad S_{A'}^A = \begin{pmatrix} I & I \\ z & \bar{z} \end{pmatrix}. \quad (6.31)$$

One can then write

$$\psi = \frac{1}{3!} \psi_{A'B'C'} \left(-ie^{K/2} e^{A'} \wedge e^{B'} \wedge e^{C'} \right) \quad (6.32)$$

where

$$\psi_{A'B'C'} = \mathcal{S}_{A'}^A \mathcal{S}_{B'}^B \mathcal{S}_{C'}^C \gamma_{ABC} \quad (6.33)$$

and

$$\mathcal{S} \equiv -ie^{-K/6} S^{-1} = -ie^{-K/6} (z - \bar{z})^{-1} \begin{pmatrix} -\bar{z} & I \\ z & -I \end{pmatrix}. \quad (6.34)$$

In this new form the amplitudes ψ_{ABC} depend on the charges and the moduli. Notice also that now we have the *same* matrix $\mathcal{S} \in \text{GL}(6, \mathbb{C})$ acting on all indices of γ_{ABC} . This reflects the fact known from the theory of quantum entanglement that the SLOCC group [14, 156] for a quantum system consisting of *indistinguishable* subsystems (now with six single particle states [20]) is represented by the same $\text{GL}(6, \mathbb{C})$ matrices acting on each entry of a tensor representing the set of amplitudes (now of a tripartite system). The basis states $-ie^{K/2} e^A \wedge e^B \wedge e^C$ for $1 \leq A < B < C \leq 6$ form an orthonormal basis with respect to the inner product of (6.13).

It is instructive to see how one recovers the *STU* case of the previous section. In particular one would like to see how the *indistinguishable* character of the subsystems represented by ψ boils down to the *distinguishable* one of the corresponding three-qubit system. In order to see this just notice that in the *STU* case we merely have 8 nonzero amplitudes to be used in (6.33). Namely we have γ_{ABC} with labels $123, 1\bar{2}\bar{3}, \dots, \bar{1}2\bar{3}, \bar{1}\bar{2}3$. Moreover, the 3×3 matrix z is now diagonal, hence the explicit form of \mathcal{S} is

$$\mathcal{S} = \frac{1}{2} e^{-K/6} \begin{pmatrix} -\bar{z}^1/y^1 & 0 & 0 & 1/y^1 & 0 & 0 \\ 0 & -\bar{z}^2/y^2 & 0 & 0 & 1/y^2 & 0 \\ 0 & 0 & -\bar{z}^3/y^3 & 0 & 0 & 1/y^3 \\ z^1/y^1 & 0 & 0 & -1/y^1 & 0 & 0 \\ 0 & z^2/y^2 & 0 & 0 & -1/y^2 & 0 \\ 0 & 0 & z^3/y^3 & 0 & 0 & -1/y^3 \end{pmatrix} \quad (6.35)$$

After switching to our usual ordering convention let us make the correspondence $321 \leftrightarrow 000, 3\bar{2}\bar{1} \leftrightarrow 001$ etc. meaning that the labels 1 and $\bar{1}$, 2 and $\bar{2}$, 3 and $\bar{3}$ refer to the labels 0 and 1 of the *first*, *second* and *third* qubit. Looking at the structure of the tensor product $\mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S}$ and recalling that $e^{-K/2} = \sqrt{8y^1 y^2 y^3}$ we quickly recover the structure of the three qubit state of (6.19). (See also the definitions of (5.125) (5.122) and (5.46).)

Let us finally comment on the structure of BPS attractors [113] in our entanglement based approach. As in the *STU* case the attractor equations require that only the $H^{3,0}$ and $H^{0,3}$ parts of the cohomology classes are non-vanishing. This implies that for our “state” of fermionic entanglement at the horizon we have

$$\psi = \psi_{321} (-ie^{K/2} e^3 \wedge e^2 \wedge e^1) - \psi_{\bar{3}\bar{2}\bar{1}} (-ie^{K/2} e^{\bar{3}} \wedge e^{\bar{2}} \wedge e^{\bar{1}}), \quad (6.36)$$

where

$$\bar{\psi}_{\bar{3}\bar{2}\bar{1}} = -\psi_{321} = Z(z_{\text{fix}}, p^0, q_0, P, Q). \quad (6.37)$$

According to the general theory [20] for classifying the SLOCC entanglement types [14] for tripartite fermionic systems with six single particle states, such attractor states belong to the fermionic generalization of the usual GHZ state well-known for

three qubits. Hence our result on the reinterpretation of the attractor mechanism as a quantum information theoretic distillation procedure in this fermionic context still holds.

In order to see this one has to solve the attractor equations [51, 113] with the result $z = x - iy$ where

$$y = \frac{1}{2}\sqrt{\mathcal{D}}(p^0 Q + P^\sharp)^{-1}, \quad x = \frac{1}{2}(2PQ - [p^0 q_0 + \text{Tr}(PQ)]I)(p^0 Q + P^\sharp)^{-1}. \quad (6.38)$$

The STU case is recovered when using diagonal matrices for $z = x - iy$, P and Q . In these expressions the definition of \mathcal{D} is

$$\mathcal{D} = -(p^0 q_0 + \text{Tr}(PQ))^2 + 4\text{Tr}(P^\sharp Q^\sharp) + 4p^0 \text{Det}Q - 4q_0 \text{Det}P \quad (6.39)$$

which is minus half of the usual quartic invariant of Freudenthal triple systems [36]. Here the Freudenthal system is the one based on the cubic Jordan algebra of 3×3 matrices with complex entries (see also section 7.7.5).

One can then show that as the result of a distillation procedure the GHZ-like state at the horizon is of the form as given by (5.138) with suitable replacements. First Cayley's hyperdeterminant D has to be replaced by its generalization \mathcal{D} as given by (6.39). Moreover, the phase α of the central charge is determined by the equation

$$\tan \alpha = \frac{p^0}{\hat{p}^0} \quad (6.40)$$

where \hat{p}^0 is the Freudenthal dual [41] of p^0 . Its explicit expression is

$$\hat{p}^0 = \frac{2\text{Det}P + p^0 \text{Tr}(PQ) + p^0 q_0}{\sqrt{-\mathcal{D}}}. \quad (6.41)$$

The stabilized states $|000\rangle$ and $|111\rangle$ of (5.138) should be replaced by their “fermionic” counterparts $(-ie^{K/2}e^3 \wedge e^2 \wedge e^1)$ and $(-ie^{K/2}e^3 \wedge e^2 \wedge e^1)$. These basis vectors should be evaluated at the stabilized values of the moduli as given by (6.38). Finally the formula for the entropy as expected is

$$S_{BH} = \pi\sqrt{\mathcal{D}}. \quad (6.42)$$

Based on our experience with the STU case where according to (2.9) the entropy formula was given in terms of a genuine tripartite measure, it is tempting to interpret $\mathcal{T}_{123} \equiv 4|\mathcal{D}|$ as an entanglement measure for three fermions with six single particle states as represented by the *normalized* state $\psi \in \wedge^3 V^*$ where $V = \mathbb{C}^6$. According to Ref. [20] within the realm of quantum information the quantity \mathcal{T}_{123} indeed works well as a basic quantity to characterize the entanglement types under the SLOCC group $\text{GL}(6, \mathbb{C})$. Within the context of black hole solutions the SLOCC group should be restricted to its real subgroup $\text{GL}(6, \mathbb{R})$. It is not difficult to see then that the different types of black holes correspond to the different entanglement types of fermionic entanglement. This correspondence runs in parallel with the original observation of Kallosh and Linde [2] that the entanglement types of three qubit states correspond to different types of STU black holes.

6.4. Other entangled systems from extra dimensions

Finally let us comment on possible generalizations. One can show [51] that the idea of the attractor mechanism as a distillation procedure works nicely also for flux attractors [78, 158]. One can study this mechanism within the context of F-theory compactifications on elliptically fibered Calabi-Yau four-folds [78]. Here the flux attractor equations are just a rephrasing of the imaginary self duality condition [158] $(\text{ISD}) \star_6 G = iG$ for the complex flux form defined as

$$G_3 = F_3 - \tau H_3 \quad (6.43)$$

where G is a combination of the type IIB NS and RR three-forms H_3 and F_3 into a new three-form G_3 which has also a dependence on a special type of new moduli

$$\tau = a + ie^{-\Phi} \quad (6.44)$$

the axion-dilaton field. The extra moduli can be incorporated into the formalism via an additional torus [51, 159, 160].

In the special case of choosing the Calabi-Yau space as the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ one finds *four-qubit systems* [51]. Indeed, it is convenient to incorporate the three-forms H and F into a complex four-form G . Then applying a correspondence similar to (6.14)-(6.15) between the Hodge diagonal basis vectors for complex four-forms [160] and the four-qubit basis vectors one can define a four-qubit state. This state depends on the quantized fluxes and the four moduli and it is of the usual form

$$|G\rangle = \mathcal{S}_4 \otimes \mathcal{S}_3 \otimes \mathcal{S}_2 \otimes \mathcal{S}_1 |g\rangle. \quad (6.45)$$

Here the extra S_4 is featuring the axion-dilaton as $z^4 \equiv \bar{\tau}$ and the explicit form of the 16 amplitudes of $|g\rangle$ is labelled as

$$\begin{pmatrix} g_{0000} & g_{0001} & g_{0010} & g_{0100} \\ g_{0111} & g_{0110} & g_{0101} & g_{0011} \end{pmatrix} = \begin{pmatrix} -p^0 & -p^1 & -p^2 & -p^3 \\ -q_0 & q_1 & q_2 & q_3 \end{pmatrix} \quad (6.46)$$

$$\begin{pmatrix} g_{1000} & g_{1001} & g_{1010} & g_{1100} \\ g_{1111} & g_{1110} & g_{1101} & g_{1011} \end{pmatrix} = \begin{pmatrix} -P^0 & -P^1 & -P^2 & -P^3 \\ -Q_0 & Q_1 & Q_2 & Q_3 \end{pmatrix}. \quad (6.47)$$

As an explicit example for this system one can revisit and reinterpret the solution found by Larsen and O'Connell [161] in the language of four-qubit entangled systems. In this special case only 8 fluxes are switched on. They are (p^0, q_1, q_2, q_3) and (Q_0, P^1, P^2, P^3) . Analysing the attractor equations, i.e. the ISD condition, one can show [161] that this configuration with 8 fluxes has a purely imaginary solution for the four moduli $z^i, i = 1, 2, 3, 4$ of the form

$$z^1 = -i \left(-\frac{Q_0 P^1 q_2 q_3}{P^2 P^3 p^0 q_1} \right)^{1/4}, \quad z^2 = -i \left(-\frac{Q_0 P^2 q_1 q_3}{P^1 P^3 p^0 q_2} \right)^{1/4} \quad (6.48)$$

$$z^3 = -i \left(-\frac{Q_0 P^3 q_1 q_2}{P^1 P^2 p^0 q_3} \right)^{1/4}, \quad z^4 = -i \left(-\frac{p^0 q_1 q_2 q_3}{Q_0 P^1 P^2 P^3} \right)^{1/4} \quad (6.49)$$

$$-\text{sgn}(Q_0 p^0) = \text{sgn}(P^1 q_1) = \text{sgn}(P^2 q_2) = \text{sgn}(P^3 q_3) = +1. \quad (6.50)$$

Recall that $z^4 = \bar{\tau}$ where $\tau = a + ie^{-\phi}$ is the axion-dilaton. Now $a = 0$ hence $-z^4$ gives the stabilized value of the dilaton. These results can be rephrased in the language of the state $|G\rangle$ as the distillation of the attractor state

$$\begin{aligned} |G\rangle_{fix} &= \frac{1}{2}(a+d)(|0000\rangle - |1111\rangle) + \frac{1}{2}(a-d)(|0011\rangle - |1100\rangle) \\ &+ \frac{1}{2}(b+c)(|0101\rangle - |1010\rangle) + \frac{1}{2}(b-c)(|0110\rangle - |1001\rangle). \end{aligned} \quad (6.51)$$

Here

$$a = i(t+z), \quad b = i(t-z), \quad c = i(y-x), \quad d = i(y+x) \quad (6.52)$$

and

$$x = \text{sgn}(q_1)\sqrt{P^1 q_1}, \quad y = \text{sgn}(q_2)\sqrt{P^2 q_2}, \quad z = \text{sgn}(q_3)\sqrt{P^3 q_3}, \quad t = -\text{sgn}(-Q_0)\sqrt{-Q_0 p^0}. \quad (6.53)$$

This state up to some phase conventions is of the same form as the normal form representative of the generic class of four-qubit entangled states [120]. This is the only entanglement class with non-vanishing hyperdeterminant, which, in this case, is an order 24 polynomial built out of the four algebraically independent invariants [45, 64]. The state $|G\rangle$ above is the result of a distillation procedure similar in character to the one discussed in the black hole context.

Note that one half the norm squared of $|G\rangle$ at the attractor point is analogous to the black hole potential in the three-qubit case. Depending merely on the fluxes at the attractor point it should be an $[\text{SL}(2)]^{\otimes 4}$ four-qubit invariant. For our example this quantity is also related to the sum of the a gravitino and chiral fermion mass squares [159]. A quick calculation shows that

$$\frac{1}{2}||G||_{fix}^2 = 2I_1 = \int F_3 \wedge H_3, \quad (6.54)$$

where I_1 is the quadratic four-qubit invariant [45].

Another interesting quantity to look at is the four-qubit generalization of Cayley's hyperdeterminant. For the definition of this $[\text{SL}(2)]^{\otimes 4}$ and permutation invariant polynomial of order 24 we refer to the literature [45, 46]. Here we just give its explicit form for our example

$$D_4 = (-Q_0 P^1 P^2 P^3)(p^0 q_1 q_2 q_3) \prod_{lkji \in (\mathbb{Z}_2)^{\times 4}} ((-1)^l t + (-1)^k z + (-1)^j y + (-1)^i x). \quad (6.55)$$

It is easy to check that $D_4 > 0$ due to our sign conventions of (6.50). A necessary condition for $D_4 \neq 0$ for this example of 8 nonvanishing fluxes is the nonvanishing of the 4 independent amplitudes of $|G\rangle_{fix}$ showing up in the 16 terms of the product.

The four-qubit hyperdeterminant D_4 also plays an important role in the structure of two-center black hole solutions of the STU model [44].

Why only tori? Clearly one should be able to remove the rather disturbing restriction to toroidal compactifications by embarking on the rich field of Calabi-Yau compactifications. Notice in this respect that the decomposition of (6.11)-(6.12) in the Hodge diagonal basis can then be used to reinterpret such formulae as *qudits* i.e. d -level systems with $d = h^{2,1} + 1$. F-theoretical flux compactifications for elliptically fibered Calabi-Yau fourfolds can then be associated with entangled systems comprising a qubit (a T^2 accounting for the axion-dilaton) and a qudit coming from a Calabi-Yau three-fold (CY_3). Alternatively, using instead of CY_3 the combination $T^2 \times K3$ we can have tripartite systems consisting of two qubits and a qudit. The idea that separable states geometrically should correspond to product manifolds and entangled ones to fibered ones was already discussed in the literature, for the simplest cases of two and three qubits. We also emphasize that though our main motivation was to account for the occurrence of qubits in these exotic scenarios we have revealed that in the string theoretical context entangled systems of more general kind than qubits should rather be considered. In particular for toroidal models we have seen that the natural arena where these systems live is the realm of fermionic entanglement [156, 157] of subsystems with *indistinguishable* parts. Of course the notion “fermionic” entanglement is simply associated with the structure of the cohomology of p -forms related to p -branes. It would be interesting to explore further consequences of these ideas in connection with the black-hole/qubit correspondence.

7. THE FREUDENTHAL TRIPLE SYSTEM CLASSIFICATION OF ENTANGLEMENT CLASSES

There is a remarkable connection between 4-dimensional supergravity theories and what have come to be known as “groups of type E_7 ” [37, 38, 41, 42, 138, 139, 162–164]. These groups may be characterized by the FTS. In particular, the $\mathcal{N} = 8$ theory is related to the FTS defined over the Jordan algebra of 3×3 Hermitian matrices, which has an $E_{7(7)}$ automorphism symmetry. Since the STU model may be embedded in the $\mathcal{N} = 8$ theory there should be a corresponding STU Jordan algebra and FTS. There is and, as we shall see, it is particularly simple.

Consequently, we would expect the elegant mathematics of Jordan algebras and Freudenthal triple systems to naturally capture the entanglement classification of three qubits. Indeed, the FTS ranks, in a succinct algebraic manner, do indeed yield the correct classification. The entanglement classes correspond to FTS ranks 0, 1, 2a, 2b, 2c, 3 and 4, or, for SLOCC* (SLOCC plus permutations), simply 0, 1, 2, 3, 4. In fact, we would argue that this is perhaps the most natural classification scheme. This is not only a matter of aesthetics. The classification of [14] based on the local entropies $S_{A,B,C}$ did not make the $SL(2, \mathbb{C})^{\otimes 3}$ symmetry manifest. While the hyperdeterminant is SLOCC-covariant, the local entropies are not; they are natural objects for a classification based on local unitaries not SLOCC. This observation is important from the perspective of generalising to n qubits. Three qubits is the only non-trivial data point we have for a full SLOCC classification. If we seek to generalise this result, we should first formulate it in terms of the objects that will generalise, i.e. SLOCC-covariants. The FTS formulation is manifestly SLOCC-covariant since the automorphism group coincides with the SLOCC-equivalence group.

More speculatively, by studying the FTS classification, one might hope to identify those algebraic features which would usefully carry over to an n -qubit generalisation. There is in fact an n -qubit generalization of the FTS, but it is not yet clear how it captures the entanglement classification [165].

7.1. Three qubit entanglement

The first SLOCC classification of three qubit entanglement was performed in [14] using a subset of the algebraically independent local unitary invariants and the 3-tangle.

The number of parameters needed to describe unnormalised LOCC (local unitary) inequivalent states is given by the dimension of the space of orbits [166],

$$\frac{\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2}{U(1) \times SU(2) \times SU(2) \times SU(2)}, \quad (7.1)$$

namely $16 - 10 = 6$. This is also equivalent to the number of algebraically independent LU invariants [167],

1. The norm squared:

$$|\Psi|^2 = \langle \Psi | \Psi \rangle \Psi. \quad (7.2)$$

2. The local entropies:

$$S_A = 4 \det \rho_A, \quad S_B = 4 \det \rho_B, \quad S_C = 4 \det \rho_C, \quad (7.3)$$

TABLE V: The values of the local entropies S_A , S_B , and S_C and the hyperdeterminant $\text{Det } a$ are used to partition three-qubit states into entanglement classes.

Class	Representative	Condition				
		Ψ	S_A	S_B	S_C	$\text{Det } a$
Null	0	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$
$A-B-C$	$ 000\rangle$	$\neq 0$	$= 0$	$= 0$	$= 0$	$= 0$
$A-BC$	$ 010\rangle + 001\rangle$	$\neq 0$	$= 0$	$\neq 0$	$\neq 0$	$= 0$
$B-CA$	$ 100\rangle + 001\rangle$	$\neq 0$	$\neq 0$	$= 0$	$\neq 0$	$= 0$
$C-AB$	$ 010\rangle + 100\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$= 0$	$= 0$
W	$ 100\rangle + 010\rangle + 001\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$= 0$
GHZ	$ 000\rangle + 111\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$

where ρ_A, ρ_B, ρ_C are the doubly reduced density matrices:

$$\rho_A = \text{Tr}_{BC} |\Psi\rangle\langle\Psi|, \quad \rho_B = \text{Tr}_{CA} |\Psi\rangle\langle\Psi|, \quad \rho_C = \text{Tr}_{AB} |\Psi\rangle\langle\Psi|. \quad (7.4)$$

3. The Kempe invariant [168]:

$$\begin{aligned} K &= \text{tr}(\rho_A \otimes \rho_B \rho_{AB}) - \text{tr}(\rho_A^3) - \text{tr}(\rho_B^3) \\ &= \text{tr}(\rho_B \otimes \rho_C \rho_{BC}) - \text{tr}(\rho_B^3) - \text{tr}(\rho_C^3) \\ &= \text{tr}(\rho_C \otimes \rho_A \rho_{CA}) - \text{tr}(\rho_C^3) - \text{tr}(\rho_A^3), \end{aligned} \quad (7.5)$$

where $\rho_{AB}, \rho_{BC}, \rho_{CA}$ are the singly reduced density matrices:

$$\rho_{AB} = \text{Tr}_C |\Psi\rangle\langle\Psi|, \quad \rho_{BC} = \text{Tr}_A |\Psi\rangle\langle\Psi|, \quad \rho_{CA} = \text{Tr}_B |\Psi\rangle\langle\Psi|. \quad (7.6)$$

4. The 3-tangle [65]:

$$\tau_{ABC} = 4|\text{Det } a_{ABC}| \quad (7.7)$$

where for the explicit expression [63, 169] see Eq.(2.3).

Hence, under local unitary operations the most general state may be written as a six real parameter generating solution [75]. For subsequent comparison with the STU black hole we restrict our attention to states with *real* coefficients a_{ABC} . In this case, one can show that there are five algebraically independent LU invariants [75]: $\text{Det } a$, S_A , S_B , S_C and the norm $\langle\Psi|\Psi\rangle\Psi$, corresponding to the dimension of

$$\frac{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2}{\text{SO}(2) \times \text{SO}(2) \times \text{SO}(2)}, \quad (7.8)$$

namely $8 - 3 = 5$. Hence, the most general real three-qubit state can be described by just five parameters [75], conveniently taken as four real numbers N_0, N_1, N_2, N_3 and an angle θ :

$$\begin{aligned} |\Psi\rangle &= -N_3 \cos^2 \theta |001\rangle - N_2 |010\rangle + N_3 \sin \theta \cos \theta |011\rangle \\ &\quad - N_1 |100\rangle - N_3 \sin \theta \cos \theta |101\rangle + (N_0 + N_3 \sin^2 \theta) |111\rangle. \end{aligned} \quad (7.9)$$

7.1.1. Entanglement classification

Dür et al. [14] used simple arguments concerning the conservation of ranks of reduced density matrices to show that there are only six types of 3-qubit equivalence classes (or seven if we count the null state); only two of which show *genuine* tripartite entanglement. They are as follows:

Null: The trivial zero entanglement orbit corresponding to vanishing states,

$$\text{Null} : \quad 0. \quad (7.10)$$

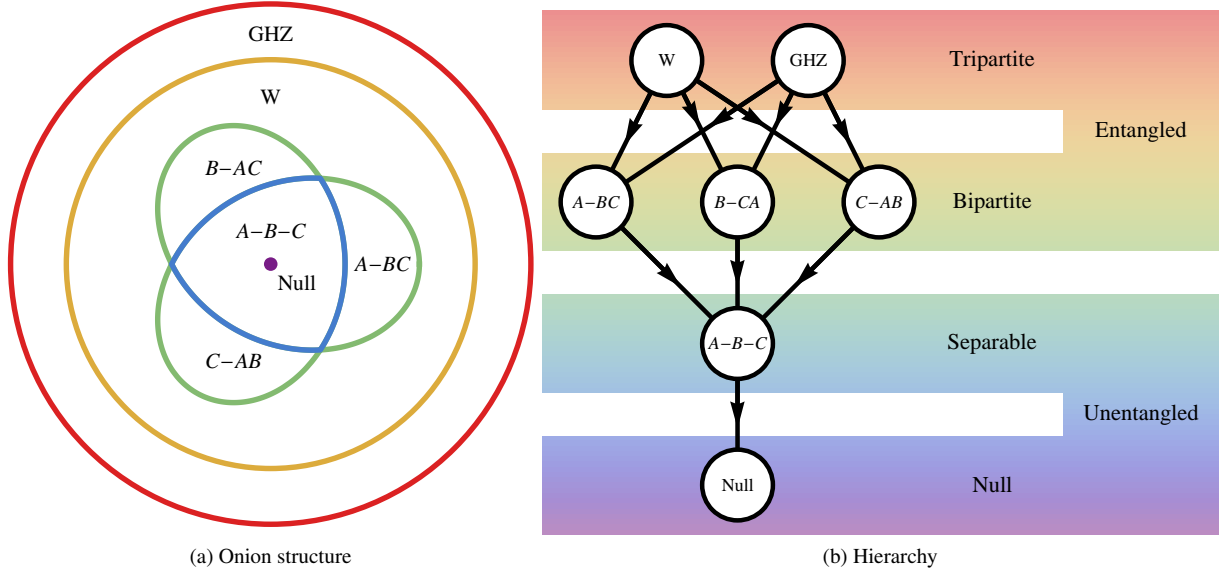


FIG. 10: (a) Onion-like classification of SLOCC orbits. (b) Stratification. The arrows are non-invertible SLOCC transformations between classes that generate the entanglement hierarchy. The partial order defined by the arrows is transitive, so we may omit e.g. $\text{GHZ} \rightarrow A-B-C$ and $A-BC \rightarrow \text{Null}$ arrows for clarity.

Separable: Another zero entanglement orbit for completely factorisable product states,

$$A-B-C : |000\rangle. \quad (7.11)$$

Biseparable: Three classes of bipartite entanglement,

$$\begin{aligned} A-BC &: |010\rangle + |001\rangle, \\ B-CA &: |100\rangle + |001\rangle, \\ C-AB &: |010\rangle + |100\rangle. \end{aligned} \quad (7.12)$$

Note, these three classes are identified under SLOCC*.

W: Three-way entangled states that do not maximally violate Bell-type inequalities in the same way as the GHZ class. However, they are robust in the sense that tracing out a subsystem generically results in a bipartite mixed state that is maximally entangled under a number of criteria [14],

$$W : |100\rangle + |010\rangle + |001\rangle. \quad (7.13)$$

GHZ: Genuinely tripartite entangled Greenberger-Horne-Zeilinger [170] states. These maximally violate Bell-type inequalities but, in contrast to class W, are fragile under the tracing out of a subsystem since the resultant state is completely unentangled,

$$\text{GHZ} : |000\rangle + |111\rangle. \quad (7.14)$$

These classes and their representative states are summarised in Table V. They are characterised [14] by the vanishing or not of the invariants listed in the table. Note that the Kempe invariant is redundant in this SLOCC classification. A visual representation of these SLOCC orbits is provided by the onion-like classification [169] of Figure 1a.

These SLOCC equivalence classes are then stratified by *non-invertible* SLOCC operations into an entanglement hierarchy [14] as depicted in Figure 1b. Note that no SLOCC operations (invertible or not) relate the GHZ and W classes; they are genuinely distinct classes of tripartite entanglement. However, from either the GHZ class or W class one may use non-invertible SLOCC transformations to descend to one of the biseparable or separable classes and hence we have a hierarchical entanglement structure. For more on three qubit entanglement see [64, 67, 171–174].

7.2. Cubic Jordan algebras

A Jordan algebra \mathfrak{J} is vector space defined over a ground field \mathbb{F} equipped with a bilinear product satisfying

$$X \circ Y = Y \circ X, \quad X^2 \circ (X \circ Y) = X \circ (X^2 \circ Y), \quad \forall X, Y \in \mathfrak{J}. \quad (7.15)$$

The class of *cubic* Jordan algebras are constructed as follows [35]. Let V be a vector space equipped with a cubic norm, i.e. a homogeneous map of degree three, $N : V \rightarrow \mathbb{F}$ s.t. $N(\lambda X) = \lambda^3 N(X)$, $\forall \lambda \in \mathbb{F}, X \in V$, with trilinear linearisation

$$N(X, Y, Z) := \frac{1}{6} [N(X + Y + Z) - N(X + Y) - N(X + Z) - N(Y + Z) + N(X) + N(Y) + N(Z)]. \quad (7.16)$$

If V further contains a base point $N(c) = 1, c \in V$ one may define the following three maps,

$$\begin{aligned} \text{Tr} : V &\rightarrow \mathbb{F}; \quad X \mapsto 3N(c, c, X), \\ S : V \times V &\rightarrow \mathbb{F}; \quad (X, Y) \mapsto 6N(X, Y, c), \\ \text{Tr} : V \times V &\rightarrow \mathbb{F}; \quad (X, Y) \mapsto \text{Tr}(X) \text{Tr}(Y) - S(X, Y). \end{aligned} \quad (7.17)$$

A cubic Jordan algebra \mathfrak{J} , with multiplicative identity $\mathbb{1} = c$, may be derived from any such vector space if N is *Jordan cubic*. That is: (i) the trace bilinear form (7.17) is non-degenerate (ii) the quadratic adjoint map, $\sharp : \mathfrak{J} \rightarrow \mathfrak{J}$, uniquely defined by $\text{Tr}(X^\sharp, Y) = 3N(X, X, Y)$, satisfies $(X^\sharp)^\sharp = N(X)X$, $\forall X \in \mathfrak{J}$. The Jordan product is then given by

$$X \circ Y = \frac{1}{2} (X \times Y + \text{Tr}(X)Y + \text{Tr}(Y)X - S(X, Y)\mathbb{1}), \quad (7.18)$$

where, $X \times Y$ is the linearisation of the quadratic adjoint, $X \times Y = (X + Y)^\sharp - X^\sharp - Y^\sharp$.

7.3. The Freudenthal triple system

Given a cubic Jordan algebra \mathfrak{J} defined over a field \mathbb{F} , one is able to construct an FTS by defining the vector space $\mathfrak{F}(\mathfrak{J})$,

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{F} \oplus \mathbb{F} \oplus \mathfrak{J} \oplus \mathfrak{J}. \quad (7.19)$$

An arbitrary element $x \in \mathfrak{F}(\mathfrak{J})$ may be written as a “ 2×2 matrix”,

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \quad \text{where } \alpha, \beta \in \mathbb{F} \quad \text{and} \quad A, B \in \mathfrak{J}. \quad (7.20)$$

The FTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product [34, 36, 162, 175, 176]:

1. Quadratic form $\{x, y\} : \mathfrak{F} \times \mathfrak{F} \rightarrow \mathbb{F}$

$$\{x, y\} = \alpha\delta - \beta\gamma + \text{Tr}(A, D) - \text{Tr}(B, C), \quad (7.21a)$$

where

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad y = \begin{pmatrix} \gamma & C \\ D & \delta \end{pmatrix}. \quad (7.21b)$$

2. Quartic form $q : \mathfrak{F} \rightarrow \mathbb{F}$

$$q(x) = -2[\alpha\beta - \text{Tr}(A, B)]^2 - 8[\alpha N(A) + \beta N(B) - \text{Tr}(A^\sharp, B^\sharp)]. \quad (7.21c)$$

3. Triple product $T : \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F} \rightarrow \mathfrak{F}$ which is uniquely defined by

$$\{T(x, y, w), z\} = q(x, y, w, z) \quad (7.21d)$$

where $q(x, y, w, z)$ is the full linearisation of $q(x)$ such that $q(x, x, x, x) = q(x)$.

The *automorphism* group is given by the set of invertible \mathbb{F} -linear transformations preserving the quadratic and quartic forms [34, 162],

$$\text{Aut}(\mathfrak{F}) = \{\sigma \in \text{Iso}_{\mathbb{F}}(\mathfrak{F}) | q(\sigma x) = q(x), \{\sigma x, \sigma y\} = \{x, y\}, \forall x, y \in \mathfrak{F}\}. \quad (7.22)$$

The automorphism group corresponds to the U-duality group of a variety 4-dimensional supergravities (see for example [134, 177] and the references therein). The conventional concept of matrix rank may be generalised to Freudenthal triple systems in a natural and $\text{Aut}(\mathfrak{F})$ invariant manner. The rank of an arbitrary element $x \in \mathfrak{F}$ is uniquely defined by [36, 176]:

$$\begin{aligned} \text{Rank} x = 1 &\Leftrightarrow 3T(x, x, y) + x\{x, y\}x = 0 \quad \forall y; \\ \text{Rank} x = 2 &\Leftrightarrow \exists y \text{ s.t. } 3T(x, x, y) + x\{x, y\}x \neq 0, \quad T(x, x, x) = 0; \\ \text{Rank} x = 3 &\Leftrightarrow T(x, x, x) \neq 0, \quad q(x) = 0; \\ \text{Rank} x = 4 &\Leftrightarrow q(x) \neq 0. \end{aligned} \quad (7.23)$$

7.4. The 3-qubit Freudenthal triple system

Definition 1 (STU cubic Jordan algebra) We define the STU cubic Jordan algebra, denoted \mathfrak{J}_{STU} , as the real vector space $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ with elements,

$$A = (A_1, A_2, A_3), \quad (7.24)$$

and cubic norm,

$$N_3(A) = A_1 A_2 A_3. \quad (7.25)$$

Using the cubic Jordan algebra construction (7.17), one finds

$$\text{Tr}(A, B) = A_1 B_1 + A_2 B_2 + A_3 B_3, \quad (7.26)$$

Then, using $\text{Tr}(A^\sharp, B) = 3N(A, A, B)$, the quadratic adjoint is given by

$$A^\sharp = (A_2 A_3, A_1 A_3, A_1 A_2), \quad (7.27)$$

and therefore

$$\begin{aligned} (A^\sharp)^\sharp &= (A_1 A_2 A_3 A_1, A_1 A_2 A_3 A_2, A_1 A_2 A_3 A_3) \\ &= N(A)A. \end{aligned} \quad (7.28)$$

It is not hard to check $\text{Tr}(A, B)$ is non-degenerate and so N_3 is Jordan cubic. Hence, we have a *bona fide* cubic Jordan algebra $\mathfrak{J}_{STU} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ with product given by

$$A \circ B = (A_1 B_1, A_2 B_2, A_3 B_3). \quad (7.29)$$

The structure and reduced structure groups are given by $[\text{SO}(2, \mathbb{R})]^3$ and $[\text{SO}(2, \mathbb{R})]^2$ respectively.

The 3-qubit cubic Jordan algebra is defined by simply promoting \mathbb{R} to \mathbb{C} ,

Definition 2 (3-qubit cubic Jordan algebra) We define the 3-qubit cubic Jordan algebra, denoted as \mathfrak{J}_{ABC} (or more generally as $\mathfrak{J}_{3\mathbb{C}}$), as the complex vector space $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ with elements:

$$A = (A_1, A_2, A_3), \quad (7.30)$$

and cubic norm,

$$N_3(A) = A_1 A_2 A_3. \quad (7.31)$$

Definition 3 (3-qubit Freudenthal triple system) We define the 3-qubit Freudenthal triple system, denoted \mathfrak{F}_{ABC} , as the complex vector space,

$$\mathfrak{F}_{ABC} := \mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{J}_{ABC} \oplus \mathfrak{J}_{ABC}, \quad (7.32)$$

with elements

$$\left(\begin{array}{c} \alpha \\ B = (B_1, B_2, B_3) \end{array} \quad A = (A_1, A_2, A_3) \quad \begin{array}{c} \\ \beta \end{array} \right). \quad (7.33)$$

We identify the eight complex components of \mathfrak{F}_{ABC} with the three qubit wavefunction $|\psi\rangle = a_{ABC}|ABC\rangle$,

$$\begin{pmatrix} \alpha & (A_1, A_2, A_3) \\ (B_1, B_2, B_3) & \beta \end{pmatrix} \leftrightarrow \begin{pmatrix} a_{111} & (a_{001}, a_{010}, a_{100}) \\ (a_{110}, a_{101}, a_{011}) & a_{000} \end{pmatrix} \quad (7.34)$$

so that

$$|\Psi\rangle = a_{ABC}|ABC\rangle \leftrightarrow \Psi := \begin{pmatrix} a_{111} & (a_{001}, a_{010}, a_{100}) \\ (a_{110}, a_{101}, a_{011}) & a_{000} \end{pmatrix}. \quad (7.35)$$

Using (7.21c) one finds that the quartic norm $\Delta(\Psi)$ is related to Cayley's hyperdeterminant by

$$\begin{aligned} \Delta(\Psi) &= \{T(\Psi, \Psi, \Psi), \Psi\} \\ &= 2 \det \gamma^A = 2 \det \gamma^B = 2 \det \gamma^C \\ &= -2 \text{Det } a_{ABC}, \end{aligned} \quad (7.36)$$

The triple product maps a state Ψ , which transforms as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $[\text{SL}(2, \mathbb{C})]^3$, to another state $T(\Psi, \Psi, \Psi)$, cubic in the state vector coefficients, also transforming as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$. Explicitly, $T(\Psi, \Psi, \Psi)$ may be written as

$$T(\Psi, \Psi, \Psi) = T_{ABC}|ABC\rangle \quad (7.37)$$

where T_{ABC} takes one of three equivalent forms

$$\begin{aligned} T_{A_3 B_1 C_1} &= \epsilon^{A_1 A_2} a_{A_1 B_1 C_1} (\gamma^A)_{A_2 A_3} \\ T_{A_1 B_3 C_1} &= \epsilon^{B_1 B_2} a_{A_1 B_1 C_1} (\gamma^B)_{B_2 B_3} \\ T_{A_1 B_1 C_3} &= \epsilon^{C_1 C_2} a_{A_1 B_1 C_1} (\gamma^C)_{C_2 C_3}. \end{aligned} \quad (7.38)$$

The γ 's are related to the local entropies of section 7.7.1 by

$$S_A = 4 \left[\text{tr } \gamma^{B\dagger} \gamma^B + \text{tr } \gamma^{C\dagger} \gamma^C \right], \quad \text{tr } \gamma^{A\dagger} \gamma^A = \frac{1}{8} [S_B + S_C - S_A] \quad (7.39)$$

and their cyclic permutations. This permits us to link T to the norm, local entropies and the Kempe invariant of section 7.7.1:

$$\langle T|T \rangle T = \frac{2}{3} (K - |\psi|^6) + \frac{1}{16} |\psi|^2 (S_A + S_B + S_C). \quad (7.40)$$

Having couched the 3-qubit system within the FTS framework we may assign an abstract FTS rank (7.23) to an arbitrary state Ψ :

$$\begin{aligned} \text{Rank } \Psi = 1 &\Leftrightarrow \Upsilon(\Psi, \Psi, \Phi) = 0, \quad \Psi \neq 0; \\ \text{Rank } \Psi = 2 &\Leftrightarrow T(\Psi) = 0, \quad \Upsilon(\Psi, \Psi, \Phi) \neq 0; \\ \text{Rank } \Psi = 3 &\Leftrightarrow \Delta(\Psi) = 0, \quad T(\Psi) \neq 0; \\ \text{Rank } \Psi = 4 &\Leftrightarrow \Delta(\Psi) \neq 0. \end{aligned} \quad (7.41)$$

Strictly speaking, the automorphism group $\text{Aut}(\mathfrak{F}_{ABC})$ is not simply $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ but includes a semi-direct product with the interchange triality $A \leftrightarrow B \leftrightarrow C$. The rank conditions are invariant under this triality. Hence the ranks naturally provide an SLOCC* classification. However, as we shall demonstrate, the set of rank 2 states may be subdivided into three distinct classes which are inter-related by this triality. In the next section we show that these rank conditions give the correct entanglement classification of three qubits as in Table VI.

7.4.1. The FTS rank entanglement classes

a. Rank 1 and the class of separable states A non-zero state Ψ is rank 1 if and only if

$$\Upsilon(\Psi, \Psi, \Phi) := 3T(\Psi, \Psi, \Phi) + \{\Psi, \Phi\}\Psi = 0, \quad \forall \Phi. \quad (7.42)$$

The weaker condition $T(\Psi, \Psi, \Psi) = 0$ implies that there is at most one non-vanishing γ since,

$$\begin{aligned} (\gamma^A)_{A_1 A_2} (\gamma^C)_{C_1 C_2} &= \epsilon^{B_1 B_2} \epsilon^{Z_1 Z_2} a_{A_1 B_1 Z_1} a_{A_2 B_2 Z_2} (\gamma^C)_{C_1 C_2} \\ &= \epsilon^{B_2 B_1} a_{A_1 B_1 C_1} T_{A_2 B_2 C_2} + \epsilon^{B_1 B_2} a_{A_2 B_2 C_1} T_{A_1 B_1 C_2}, \end{aligned} \quad (7.43)$$

Class	Rank	FTS rank condition	
		vanishing	non-vanishing
Null	0	Ψ	—
$A-B-C$	1	$3T(\Psi, \Psi, \Phi) + \{\Psi, \Phi\}\Psi$	Ψ
$A-BC$	2a	$T(\Psi, \Psi, \Psi)$	γ^A
$B-CA$	2b	$T(\Psi, \Psi, \Psi)$	γ^B
$C-AB$	2c	$T(\Psi, \Psi, \Psi)$	γ^C
W	3	$\Delta(\Psi)$	$T(\Psi, \Psi, \Psi)$
GHZ	4	—	$\Delta(\Psi)$

TABLE VI: The entanglement classification of three qubits as according to the FTS rank system.

and similarly for $(\gamma^B)_{B_1 B_2}(\gamma^A)_{A_1 A_2}$ and $(\gamma^C)_{C_1 C_2}(\gamma^B)_{B_1 B_2}$. In component form Υ is given by,

$$\begin{aligned}
-\Upsilon_{A_1 B_1 C_1} = & \epsilon^{A_2 A_3} b_{A_3 B_1 C_1} (\gamma^A)_{A_1 A_2} \\
& + \epsilon^{B_2 B_3} b_{A_1 B_3 C_1} (\gamma^B)_{B_1 B_2} \\
& + \epsilon^{C_2 C_3} b_{A_1 B_1 C_3} (\gamma^C)_{C_1 C_2}
\end{aligned} \tag{7.44}$$

where

$$|\phi\rangle = b_{ABC}|ABC\rangle \leftrightarrow \Phi = \begin{pmatrix} b_{111} & (b_{001}, b_{010}, b_{100}) \\ (b_{110}, b_{101}, b_{011}) & b_{000} \end{pmatrix}. \tag{7.45}$$

Hence, (7.42) implies all three gammas must vanish. Using (7.39) it is then clear that all three local entropies vanish.

Conversely, $S_A = S_B = S_C = 0$ implies that each of the three γ 's vanish and the rank 1 condition is satisfied. Hence FTS rank 1 is equivalent to the class of separable states as in Table VI.

b. Rank 2 and the class of biseparable states A nonzero state Ψ is rank 2 or less if and only if $T(\Psi, \Psi, \Psi) = 0$, which implies there is at most one non-vanishing γ . To be rank > 1 there must exist some Φ such that $3T(\Psi, \Psi, \Phi) + \{\Psi, \Phi\}\Psi \neq 0$, which implies there is at least one non-vanishing γ . Hence, rank 2 states have precisely one nonzero γ .

Using (7.39) it is clear that the choices $\gamma^A \neq 0$ or $\gamma^B \neq 0$ or $\gamma^C \neq 0$ give $S_A = 0, S_{B,C} \neq 0$ or $S_B = 0, S_{C,A} \neq 0$ or $S_C = 0, S_{A,B} \neq 0$, respectively. These are precisely the conditions for the biseparable class $A-BC$ or $B-CA$ or $C-AB$ presented in Table V.

Conversely, using (7.39) and the fact that the local entropies and $\text{tr}(\gamma^\dagger \gamma)$ are positive semidefinite, we find that all states in the biseparable class are rank 2, the particular subdivision being given by the corresponding non-zero γ . Hence FTS rank 2 is equivalent to the class of biseparable states as in Table VI.

c. Rank 3 and the class of W-states A non-zero state Ψ is rank 3 if $\Delta(\Psi) = -2 \text{Det } a = 0$ but $T(\Psi, \Psi, \Psi) \neq 0$. From (7.38) all three γ 's are then non-zero but from (7.36) all have vanishing determinant. In this case (7.39) implies that all three local entropies are non-zero but $\text{Det } a = 0$. So all rank 3 Ψ belong to the W-class.

Conversely, from (7.39) it is clear that no two γ 's may simultaneously vanish when all three S 's > 0 . We saw in section 7.4.7.4.1 that $T(\Psi, \Psi, \Psi) = 0$ implied at least two of the γ 's vanish. Consequently, for all W-states $T(\Psi, \Psi, \Psi) \neq 0$ and, therefore, all W-states are rank 3. Hence FTS rank 3 is equivalent to the class of W-states as in Table VI.

d. Rank 4 and the class of GHZ-states The rank 4 condition is given by $\Delta(\Psi) \neq 0$ and, since for the 3-qubit FTS $\Delta(\Psi) = -2 \text{Det } a$, we immediately see that the set of rank 4 states is equivalent to the GHZ class of genuine tripartite entanglement as in Table VI.

Note, $\text{Aut}(\mathfrak{F}_{ABC})$ acts transitively only on rank 4 states with the same value of $\Delta(\Psi)$ as in the standard treatment. The GHZ class really corresponds to a 1-dimensional space of orbits parametrised by Δ .

In summary, we have demonstrated that each rank corresponds to one of the entanglement classes described in section 7.7.1.

7.4.2. SLOCC orbits

We now turn our attention to the coset parametrisation of the entanglement classes. The coset space of each orbit ($i = 1, 2, 3, 4$) is given by G/H_i where $G = [\text{SL}(2, \mathbb{C})]^3$ is the SLOCC group and $H_i \subset [\text{SL}(2, \mathbb{C})]^3$ is the stability subgroup leaving the representative state of the i th orbit invariant. We proceed by considering the infinitesimal action of $\text{Aut}(\mathfrak{F})$ on the representative states of each class. The subalgebra annihilating the representative state gives, upon exponentiation, the stability group H . Note, $\text{der}(\mathfrak{J}_{ABC})$ is empty due to the associativity of \mathfrak{J}_{ABC} . Consequently, $\mathfrak{Str}(\mathfrak{J}_{ABC}) = L_1 \mathbb{C} \oplus \mathfrak{Str}_0(\mathfrak{J}_{ABC})$ has complex dimension 3, while $\mathfrak{Str}_0(\mathfrak{J}_{ABC})$ is now simply $L_{\mathfrak{J}'}$ and has complex dimension 2. Recall, $\mathfrak{Str}(\mathfrak{J}_{ABC})$ and $\mathfrak{Str}_0(\mathfrak{J}_{ABC})$ generate $[\text{SO}(2, \mathbb{C})]^3$ and $[\text{SO}(2, \mathbb{C})]^2$, respectively the structure and reduced structure groups of $\mathfrak{J}_{\mathbb{C}}$.

Class	Orbits	dim	Projective orbits	dim
Separable	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{[\mathrm{SO}(2, \mathbb{C})]^2 \ltimes \mathbb{C}^3}$	4	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{[\mathrm{SO}(2, \mathbb{C}) \ltimes \mathbb{C}]^3}$	3
Biseparable	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{\mathrm{O}(3, \mathbb{C}) \times \mathbb{C}}$	5	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{\mathrm{O}(3, \mathbb{C}) \times (\mathrm{SO}(2, \mathbb{C}) \ltimes \mathbb{C})}$	4
W	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{\mathbb{C}^2}$	7	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{\mathrm{SO}(2, \mathbb{C}) \ltimes \mathbb{C}^2}$	6
GHZ	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{[\mathrm{SO}(2, \mathbb{C})]^2}$	7	$\frac{[\mathrm{SL}(2, \mathbb{C})]^3}{[\mathrm{SO}(2, \mathbb{C})]^2}$	7

TABLE VII: Coset spaces of the orbits of the 3-qubit state space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ under the action of the SLOCC group $[\mathrm{SL}(2, \mathbb{C})]^3$.

Class	FTS Rank	$\Delta(\Psi)$	Orbits	dim
Separable	1	$= 0$	$\frac{[\mathrm{SL}(2, \mathbb{R})]^3}{[\mathrm{SO}(1, 1)]^2 \ltimes \mathbb{R}^3}$	4
Biseparable	2	$= 0$	$\frac{[\mathrm{SL}(2, \mathbb{R})]^3}{\mathrm{O}(2, 1) \times \mathbb{R}}$	5
W	3	$= 0$	$\frac{[\mathrm{SL}(2, \mathbb{R})]^3}{\mathbb{R}^2}$	7
GHZ	4	< 0	$\frac{[\mathrm{SL}(2, \mathbb{R})]^3}{[\mathrm{SO}(1, 1)]^2}$	7
GHZ	4	> 0	$\frac{[\mathrm{SL}(2, \mathbb{R})]^3}{[\mathrm{U}(1)]^2}$	7
GHZ	4	> 0	$\frac{[\mathrm{SL}(2, \mathbb{R})]^3}{[\mathrm{U}(1)]^2}$	7

TABLE VIII: Coset spaces of the orbits of the real case $\mathfrak{J}_{3\mathbb{R}} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ under $[\mathrm{SL}(2, \mathbb{R})]^3$.

The results are summarised in Table VII. To be clear, in the preceding analysis we have regarded the three-qubit state as a point in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the philosophy adopted in, for example, [166, 167, 178]. We could have equally well considered the projective Hilbert space regarding states as rays in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, that is, identifying states related by a global complex scalar factor, as was done in [67, 169, 179]. The coset spaces obtained in this case are also presented in Table VII, the dimensions of which agree with the results of [64, 169]. Note that the three-qubit separable projective coset is just a direct product of three individual qubit cosets $\mathrm{SL}(2, \mathbb{C})/\mathrm{SO}(2, \mathbb{C}) \ltimes \mathbb{C}$. Furthermore, the biseparable projective coset is just the direct product of the two entangled qubits coset $[\mathrm{SL}(2, \mathbb{C})]^2/\mathrm{O}(3, \mathbb{C})$ and an individual qubit coset.

As noted in [3, 180], the case of real qubits or “rebits” is qualitatively different from the complex case. An interesting observation is that on restricting to real states the GHZ class actually has two distinct orbits, characterised by the sign of $\Delta(\Psi)$. This difference shows up in the cosets in the different possible real forms of $[\mathrm{SO}(2, \mathbb{C})]^2$. For positive $\Delta(\Psi)$ there are two disconnected orbits, both with $[\mathrm{SL}(2, \mathbb{R})]^3/[\mathrm{U}(1)]^2$ cosets, while for negative $\Delta(\Psi)$ there is one orbit $[\mathrm{SL}(2, \mathbb{R})]^3/[\mathrm{SO}(1, 1, \mathbb{R})]^2$. In which of the two positive $\Delta(\Psi)$ orbits a given state lies is determined by the sign of the eigenvalues of the three γ ’s, as shown in Table VIII. This phenomenon also has its counterpart in the black-hole context [28, 59, 89, 134, 141, 181], where the two disconnected $\Delta(\Psi) > 0$ orbits are given by 1/2-BPS black holes and non-BPS black holes with vanishing central charge respectively [134].

7.5. Fermionic entanglement classification from the Freudenthal triple system

The 3-qubit Freudenthal triple system can be generalised to classify the entanglement of several other multipartite systems simply by considering different Jordan algebras. In particular, the FTS naturally captures the structure of entanglement for various multipartite systems of both distinguishable and *indistinguishable* constituents [20, 182]. While there is a significant body of literature on the entanglement classification of distinguishable multipartite systems, much less is known for indistinguishable constituents. For bipartite fermionic and bosonic systems a number of useful results exist [183–188]. For example, a two-fermion systems with $2k$ single-particle states admits a Schmidt-like decomposition [187, 188].

To differentiate the various possible distinguishable and indistinguishable (bosonic/fermionic) systems we will denote the basis vectors for a k -level distinguishable, indistinguishable-bosonic and indistinguishable-fermionic system by, e_i , b_i and f_i ,

Space $\mathfrak{F}(\mathcal{H})$	$\text{Aut}(\mathfrak{F})$	Class	Representatives
$\bigwedge^3 \mathbb{C}^6$	$\text{SL}(6, \mathbb{C})$	GHZ	$\frac{1}{\sqrt{2}}(f_1 \wedge f_2 \wedge f_3 + f_4 \wedge f_5 \wedge f_6)$
		W	$\frac{1}{\sqrt{3}}(f_4 \wedge f_2 \wedge f_3 + f_1 \wedge f_5 \wedge f_3 + f_1 \wedge f_2 \wedge f_6)$
		Biseparable	$\frac{1}{\sqrt{2}}(f_1 \wedge f_2 \wedge f_3 + f_1 \wedge f_5 \wedge f_6)$
		Separable	$f_1 \wedge f_2 \wedge f_3$
$\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^4$	$\text{SL}(2, \mathbb{C}) \times \text{SL}(4, \mathbb{C})$	GHZ	$\frac{1}{\sqrt{2}}(e_0 \otimes (f_0 \wedge f_1) + e_1 \otimes (f_2 \wedge f_3))$
		W	$\frac{1}{\sqrt{3}}(e_0 \otimes (f_2 \wedge f_3) + e_1 \otimes (f_0 \wedge f_3) + e_1 \otimes (f_2 \wedge f_1))$
		Biseparable (a)	$\frac{1}{\sqrt{2}}e_0 \otimes (f_0 \wedge f_1 + f_2 \wedge f_3)$
		Biseparable (b)	$\frac{1}{\sqrt{2}}(e_0 \otimes (f_0 \wedge f_1) + e_1 \otimes (f_0 \wedge f_3))$
		Separable	$e_0 \otimes (f_0 \wedge f_1)$
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	$\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$	GHZ	$\frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)$
		W	$\frac{1}{\sqrt{3}}(e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1)$
		Biseparable (a)	$\frac{1}{\sqrt{2}}(e_0 \otimes (e_0 \otimes e_0 + e_1 \otimes e_1))$
		Biseparable (b)	$\frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1)$
		Biseparable (c)	$\frac{1}{\sqrt{2}}((e_0 \otimes e_0 + e_1 \otimes e_1) \otimes e_0)$
$\mathbb{C}^2 \otimes \text{Sym}^2(\mathbb{C}^2)$	$\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$	Separable	$e_0 \otimes e_0 \otimes e_0$
		GHZ	$\frac{1}{\sqrt{2}}(e_0 \otimes (b_0 \otimes b_0) + e_1 \otimes (b_1 \otimes b_1))$
		W	$\frac{1}{\sqrt{3}}(e_1 \otimes (b_0 \otimes b_0) + e_0 \otimes (b_1 \otimes b_0 + b_0 \otimes b_1))$
		Biseparable (a)	$\frac{1}{\sqrt{2}}(e_0 \otimes (b_0 \otimes b_0 + b_1 \otimes b_1))$
		Separable	$e_0 \otimes (b_0 \otimes b_0)$
$\text{Sym}^3(\mathbb{C}^2)$	$\text{SL}(2, \mathbb{C})$	GHZ	$\frac{1}{\sqrt{2}}(b_0 \otimes b_0 \otimes b_0 + b_1 \otimes b_1 \otimes b_1)$
		W	$\frac{1}{\sqrt{3}}(b_1 \otimes b_0 \otimes b_0 + b_0 \otimes b_1 \otimes b_0 + b_0 \otimes b_0 \otimes b_1)$
		Separable	$b_0 \otimes b_0 \otimes b_0$

TABLE IX: Representatives of SLOCC orbits of quantum mechanical systems classified via Freudenthal's construction [182].

where $i = 1, \dots, k$, respectively. The n -partite basis vectors are then given by,

$$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n}, \quad b_{(i_1} \otimes b_{i_2} \otimes \dots \otimes b_{i_n}), \quad f_{i_1} \wedge f_{i_2} \wedge \dots \wedge f_{i_n}. \quad (7.46)$$

The two simplest examples we consider are essentially degenerations of the 3-qubit system $\mathfrak{F}(\mathfrak{J}_{3\mathbb{C}})$, given by replacing $\mathfrak{J}_{3\mathbb{C}}$ with $\mathfrak{J}_{2\mathbb{C}} = \mathbb{C} \oplus \mathbb{C}$ and $\mathfrak{J}_{\mathbb{C}} = \mathbb{C}$. The cubic norms of $\mathfrak{J}_{2\mathbb{C}}$ and $\mathfrak{J}_{\mathbb{C}}$ are given by setting $A_2 = A_3$ and $A_1 = A_2 = A_3$, respectively,

$$N_{\mathfrak{J}_{2\mathbb{C}}}(A) = A_1 A_2^2, \quad N_{\mathfrak{J}_{3\mathbb{C}}}(A) = A_1^3. \quad (7.47)$$

Since $\mathfrak{J}_{\mathbb{C}} \subset \mathfrak{J}_{2\mathbb{C}} \subset \mathfrak{J}_{3\mathbb{C}}$, clearly $\mathfrak{F}(\mathfrak{J}_{\mathbb{C}}) \subset \mathfrak{F}(\mathfrak{J}_{2\mathbb{C}}) \subset \mathfrak{F}(\mathfrak{J}_{3\mathbb{C}})$. As vector spaces $\mathfrak{F}(\mathfrak{J}_{\mathbb{C}})$ and $\mathfrak{F}(\mathfrak{J}_{2\mathbb{C}})$ are given by $\text{Sym}^3(\mathbb{C}^2)$ and $\mathbb{C}^2 \otimes \text{Sym}^2(\mathbb{C}^2)$, respectively. The former is the Hilbert space of three indistinguishable bosonic qubits, while the later corresponds to the Hilbert space of a single distinguishable qubit and two indistinguishable bosonic qubits. Using the identification between $\Psi \in \mathfrak{F}(\mathfrak{J}_{3\mathbb{C}})$ and $a_{ABC}|ABC\rangle$ given in (7.35), elements in $\mathfrak{F}(\mathfrak{J}_{2\mathbb{C}})$ and $\mathfrak{F}(\mathfrak{J}_{\mathbb{C}})$ can be written in terms of the appropriate symmetrizations, i.e. $a_{A(BC)}$ and $a_{(ABC)}$, respectively. The antisymmetric bilinear form, quartic norm and triple product are then simply given by symmetrizing those of $\mathfrak{F}(\mathfrak{J}_{3\mathbb{C}})$. See [139, 182] for details.

The automorphism (SLOCC) groups are given by $\text{SL}(2, \mathbb{C})_A \times \text{SL}(2, \mathbb{C})_{(BC)}$ and $\text{SL}(2, \mathbb{C})_{(ABC)}$, under which $a_{A(BC)}$ and $a_{(ABC)}$ transform as a $(2, 3)$ and a 4 , respectively. Just as in the conventional 3-qubit case, the orbit ("entanglement") classification is given by the FTS ranks defined in (7.23). The classes/orbits and their representative states are given in Table IX [182]. These essentially have the same structure as the 3-qubit case. One subtlety is that the biseparable class is absent for $\mathfrak{F}(\mathfrak{J}_{\mathbb{C}})$. This is a consequence of $N_{\mathfrak{J}_{3\mathbb{C}}}(A) = 0 \Rightarrow A = 0$. The corresponding orbit classification over the reals is given in [139]. The real case underlies the structure of the ST^2 and T^3 models of $\mathcal{N} = 2$ supergravity coupled to two and one vector multiplets respectively. The analysis of the single-centre extremal black hole solutions in these models, via the orbits, is given in [28, 103, 114, 134, 138, 189].

These bosonic systems were embedded in the 3-qubit FTS. Going the other way, i.e. embedding the 3-qubit system in larger FTSs, we can accommodate a 3-fermion system. As pointed out in [20] the first non-trivial 3-fermion system has 6-level constituents, since the 5-level and 4-level cases may be mapped to 2- and 1-fermion systems, respectively, using $\bigwedge^3 \mathbb{C}^5 \cong \bigwedge^2 \mathbb{C}^5$ and $\bigwedge^3 \mathbb{C}^4 \cong \bigwedge^1 \mathbb{C}^4$. The system of three 6-level fermions has Hilbert space $\bigwedge^3 \mathbb{C}^6$.

Let $\{f_1, f_2, f_3, f_{\bar{1}} \equiv f_4, f_{\bar{2}} \equiv f_5, f_{\bar{3}} \equiv f_6\}$ be an orthonormal basis of \mathbb{C}^6 , and let $x \wedge y \wedge z$ denote the normalized wedge product of the vectors $x, y, z \in \mathbb{C}^6$:

$$x \wedge y \wedge z = \frac{1}{\sqrt{6}}(x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y - x \otimes z \otimes y - z \otimes y \otimes x - y \otimes x \otimes z) \quad (7.48)$$

Then a generic (unnormalised) 3-fermion state may be written as

$$|P\rangle = \sum_{1 \leq a < b < c \leq 3} P_{abc} f_a \wedge f_b \wedge f_c. \quad (7.49)$$

The state space coincides with the FTS defined over the Jordan algebra $H_3(\tilde{\mathbb{C}}_{\mathbb{C}})$ of 3×3 Hermitian matrices over the complexified split-complexes [36, 190], or, more simply, the Jordan algebra $M_3(\mathbb{C})$ of 3×3 complex matrices [20],

$$\mathfrak{F}_{3\text{-fermion}} = \mathbb{C} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}). \quad (7.50)$$

The cubic norm on $M_3(\mathbb{C})$ is simply given by the determinant and the remaining Jordan/FTS structures then follow from the definitions given in section 7.7.2 and section 7.7.3.

Explicitly, for $x = (\alpha, \beta, A, B) \in \mathfrak{F}_{3\text{-fermion}}$ we have

$$\alpha = P_{123} \quad \beta = P_{\bar{1}\bar{2}\bar{3}} \quad A = \begin{bmatrix} P_{1\bar{2}\bar{3}} & P_{1\bar{3}\bar{1}} & P_{1\bar{1}\bar{2}} \\ P_{2\bar{2}\bar{3}} & P_{2\bar{3}\bar{1}} & P_{2\bar{1}\bar{2}} \\ P_{3\bar{2}\bar{3}} & P_{3\bar{3}\bar{1}} & P_{3\bar{1}\bar{2}} \end{bmatrix} \quad B = \begin{bmatrix} P_{123} & P_{131} & P_{112} \\ P_{223} & P_{231} & P_{212} \\ P_{323} & P_{331} & P_{312} \end{bmatrix}. \quad (7.51)$$

Note, the conventional 3-qubit system is given by the sub Jordan algebra of diagonal matrices in $M_3(\mathbb{C})$, in which case the quartic norm reduces to Cayley's hyperdeterminant [20].

The automorphism group is given by $\text{SL}(6, \mathbb{C})$, which partitions the state space into four orbits given precisely by the ranks, as in the 3-qubit case. Of course, the GHZ class is actually a 1-dimensional space of orbits parametrised by the quartic norm, which collapses to a single orbit under $\text{GL}(6, \mathbb{C})$. The representative states are given Table IX. Finally, there is an intermediate FTS sitting between $\mathfrak{F}_{3\mathbb{C}}$ and $\mathfrak{F}_{3\text{-fermion}}$, given by the Jordan algebra $\mathbb{C} \oplus M_2(\mathbb{C})$, with cubic norm $a \det(A)$ for $a \in \mathbb{C}, A \in M_2(\mathbb{C})$, which describes a single distinguishable qubit with two 4-level fermions [182]. The entanglement classes are given in Table IX and further details may be found in [182].

8. CONCLUSIONS

Our purpose in this review has been to highlight the numerous (and still growing) useful ways in which black holes in string theory can inform qubit entanglement in QIT and vice-versa, whether or not there is physical connection between the two. These include not only the originally discovered relations between the U-duality invariant entropies of the black holes and the corresponding entanglement measures in QIT, and the more recent work on the attractor mechanism and concurrence and classifying four qubit entanglement and Freudenthal triple systems, but also less familiar results on the relation to Hamming codes, geometric hyperplanes and Mermin Squares. All this suggests that there is till much more to be uncovered.

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- [191] The condition that a measure be monotonically decreasing *on average* under LOCC is actually a stronger condition.
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- [193] Indeed, failure would call into question the usefulness of the SLOCC entanglement classification.